Magnetic Energy Density, Ferromagnetism, and the Magnet-on-Wall Problem

PHYS 4392 — Electromagnetism | Lecture of April 27, 2026

1. Energy Stored in the Coaxial-Cable Field

Last lecture we found $B(s)$ for the coaxial cable in three regions. To get the total stored magnetic energy — and hence the inductance — we integrate the energy density over all space:

$$U = \int u(\vec{r})\, dV = \int \frac{B(\vec{r})^2}{2\mu_0}\, dV\,.$$

By the cylindrical symmetry of the coax, this integral splits cleanly into three independent pieces, one along each cylindrical coordinate:

$$U = \int_0^L dz \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{B(s)^2}{2\mu_0}\, s\, ds\,.$$

The z integral gives the cable length L; the φ integral gives 2π. Only the radial integral does any real work.

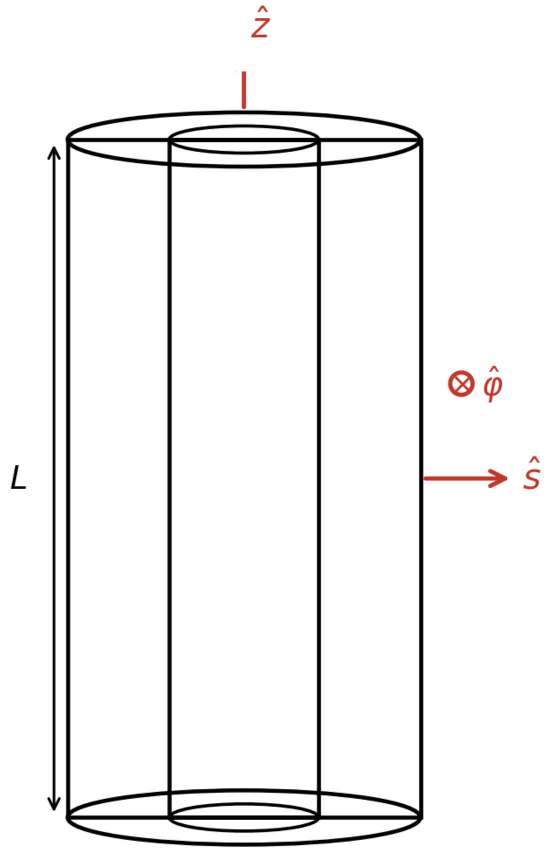

Figure 1. Side view of the coaxial cable showing the cylindrical coordinate frame. The magnetic field circulates azimuthally (φ̂, into the page on the indicated side); the energy density depends only on s.

Profile of B(s) for the solid-inner case

For the solid inner conductor of radius a with outer shell at b, the field grows linearly inside the conductor, falls as $1/s$ in the gap, and vanishes outside the shield:

$$B(s) \propto \frac{s}{a^2}\,(s<a), \qquad B(s) \propto \frac{1}{s}\,(a<s<b), \qquad B(s) = 0\,(s>b)\,.$$

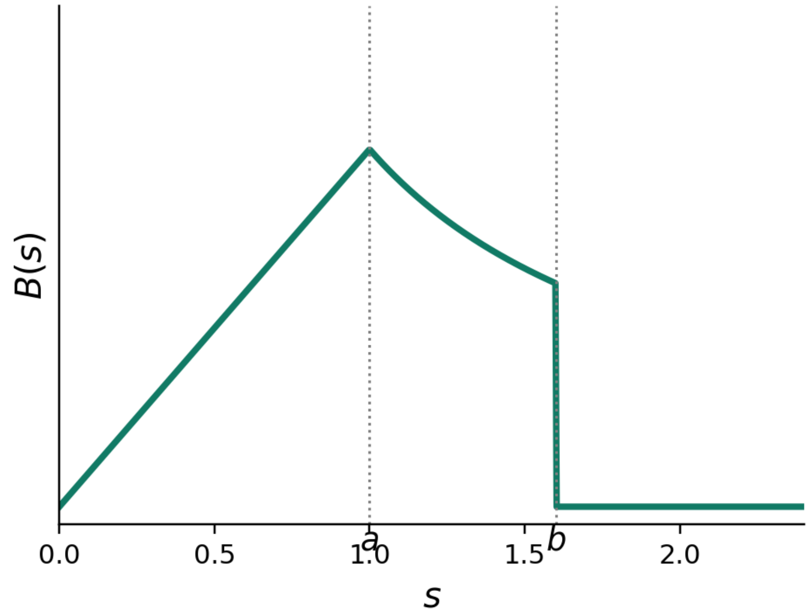

Figure 2. The radial profile B(s): linear ramp inside the solid conductor, 1/s tail in the gap, and zero outside. The energy integrand B^2s determines how each region contributes to U.

Plugging in and carrying out the radial integral for the gap region $a < s < b$ (the dominant contribution),

$$U_{\text{gap}} = \frac{\mu_0 I^2 L}{4\pi} \ln\left(\frac{b}{a}\right),$$

and equating this to $U = (1/2)LI^2$ identifies the inductance per unit length:

$$\frac{L_{\text{coax}}}{L} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right).$$

The logarithm is characteristic of cylindrical geometries. If you ever see $\mu_0/(2\pi)\ln(b/a)$ in a result, a coax was involved somewhere in the derivation.

2. The Quadrupole Term

Returning to the multipole expansion, after the monopole and dipole come the quadrupole and higher moments. The quadrupole correction takes the form

$$V_{\text{quad}}(\vec{r}) \sim \sum_{ij} M_{ij} \frac{\partial^2}{\partial x_i\,\partial x_j}\left(\frac{1}{r}\right),$$

where M_{ij} is the quadrupole moment tensor of the source. The second derivatives of $1/r$ give the characteristic angular structure,

$$\frac{\partial^2}{\partial x_i\,\partial x_j}\left(\frac{1}{r}\right) = \frac{1}{r^5}\left(3x_i x_j - r^2\delta_{ij}\right),$$

so the quadrupole potential decays as $1/r^3$, with an angular pattern that is symmetric and traceless. The take-home: if both the monopole and dipole vanish, the quadrupole is the leading nonzero term, and its angular dependence is more elaborate than the simple $p \cdot \hat{r}$ of the dipole.

3. Ferromagnetism and Hysteresis

The magnetization M is the magnetic dipole moment per unit volume. For most materials — paramagnets and diamagnets — M is linear in H: $\mathbf{M} = \chi_m \mathbf{H}$, with a susceptibility tensor χ_m that need not be diagonal. Ferromagnets behave very differently: M depends on the entire history of the applied field, not just its current value. We say ferromagnets have *memory*.

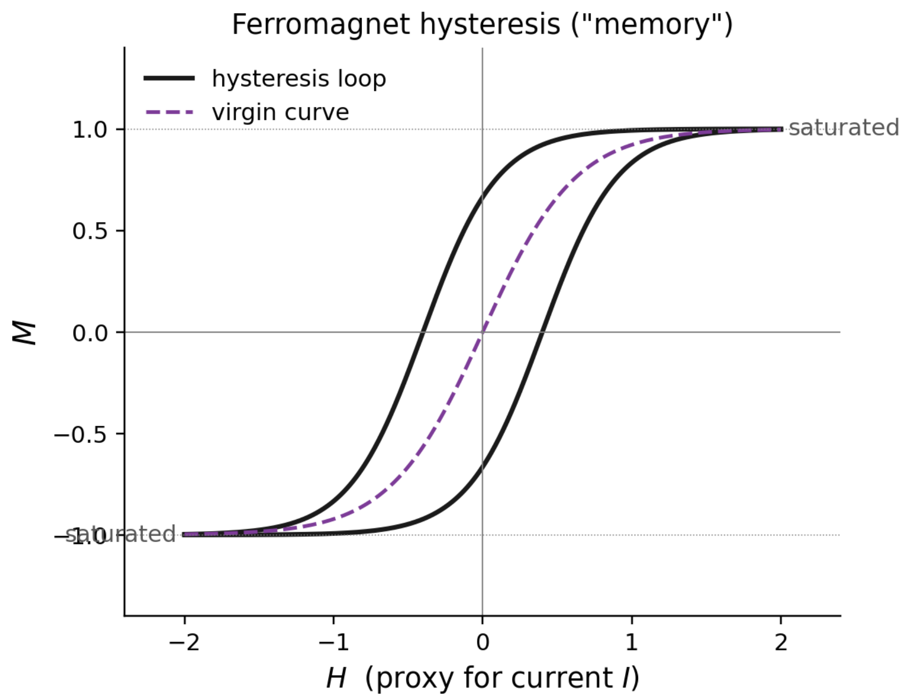

Figure 3. The hysteresis loop of a ferromagnet. The dashed virgin curve shows the response of an initially unmagnetized sample; once driven into saturation, the system traces out a closed loop on subsequent reversals, retaining nonzero M at H = 0 (the remanence).

Linear regimes — soft ferro, para, dia

Setting hysteresis aside, the linear approximation $\mathbf{M} = \chi H$ separates magnetic materials into three regimes by the sign and magnitude of χ:

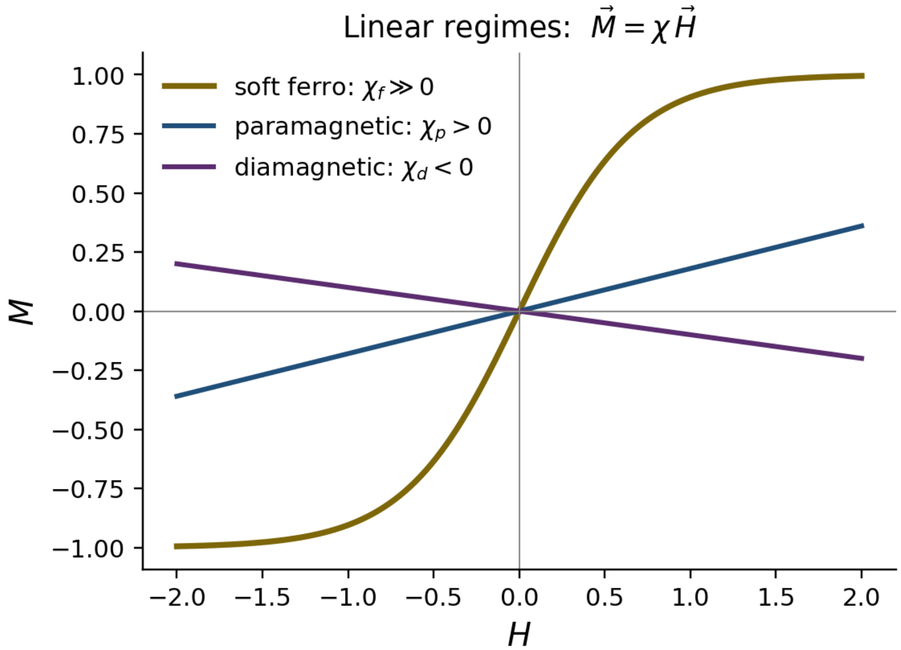

Figure 4. Three linear regimes. Soft ferromagnets have very large positive susceptibility $\chi_f \sim 10^4$–10^5 in MKS units. Paramagnets have small positive $\chi_p > 0$ (aligned moments). Diamagnets have small negative $\chi_d < 0$ (Lenz-law response).

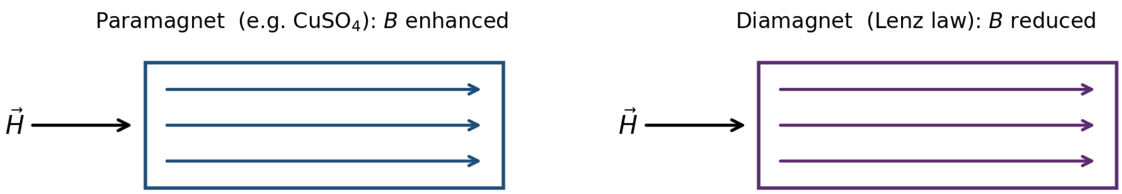

*Figure 5. The microscopic distinction. In a paramagnet (e.g. $CuSO_4$) the unpaired moments align with **H** and enhance **B** inside the material. In a diamagnet, induced currents oppose the change by Lenz's law and slightly reduce **B** inside.*

Hard ferromagnets

A *hard* ferromagnet is one whose magnetization is essentially fixed: **M** ≈ constant, independent of the applied H over normal operating ranges. The relationship **M** = χ**H** simply does not apply — one cannot solve for χ because M does not vary with H. The idealized M-H curve looks rectangular: two horizontal saturated branches connected by very steep transitions far from equilibrium.

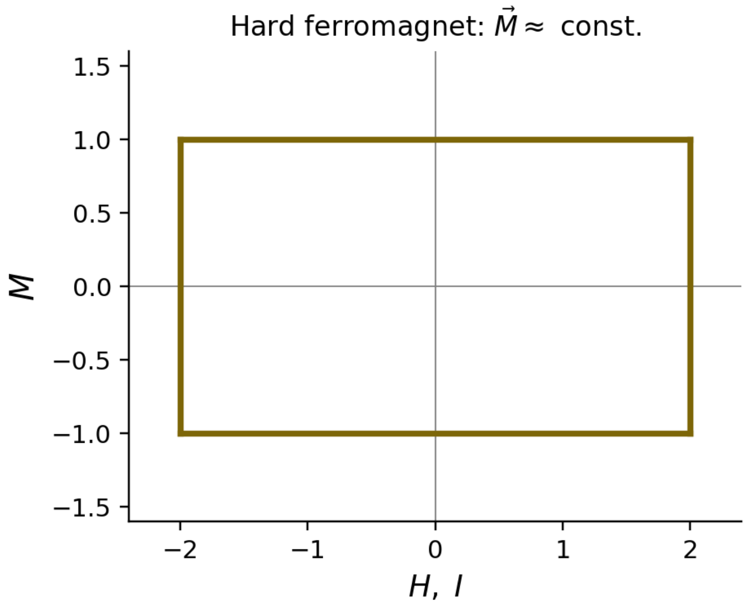

Figure 6. Idealized hard-ferromagnet M-H curve. The magnetization is essentially $\pm M_{sat}$ across the entire operating range; the linear regime $\mathbf{M} = \chi\mathbf{H}$ is not a useful description here.

4. A Hard Ferromagnet on an Iron Wall

A practical question: a long thin bar magnet (length ℓ much greater than $\sqrt{A}$, where A is the cross-sectional area) is held against an iron (Fe) wall. With what force does it stick? The magnetization is the standard hard-ferromagnet value, $M \sim 10^6$ A/m. We take **M** aligned with the bar's long axis.

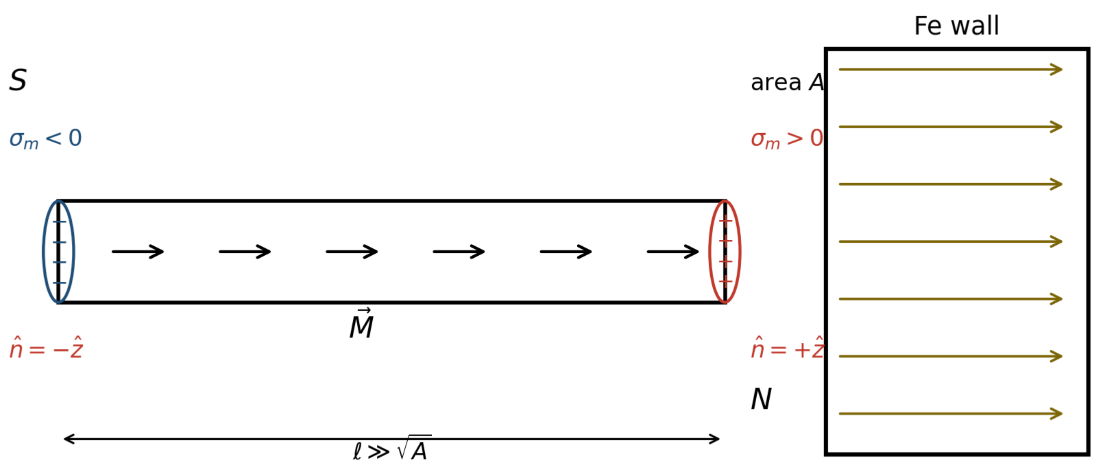

Figure 7. A bar magnet of length ℓ held against an iron wall. The S and N faces carry fictitious surface charge densities $\sigma_m = \hat{n}\cdot\mathbf{M}$, with $\hat{n}$ the outward normal at each face.

Two equivalent pictures

There are two complementary ways to describe a permanent magnet, and the magnet-on-wall problem makes the choice strategic.

Picture A: bound surface currents

The actual microscopic source of **B** outside the magnet is the bound current density on its surface,

$$\vec{K}_{\text{bound}} = \vec{M} \times \hat{n},$$

which circulates around the bar like the windings of a solenoid — indeed, a uniformly magnetized rod is field-equivalent to a long thin solenoid.

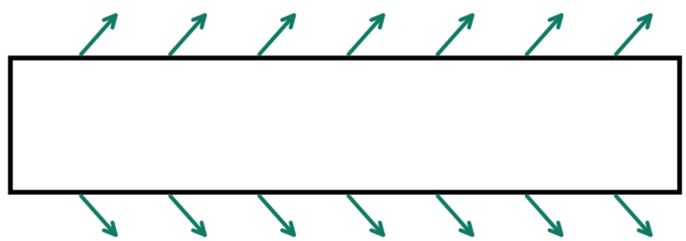

Figure 8. Bound surface current $\mathbf{K}_{bound} = \mathbf{M}\times\hat{n}$ circulating around the bar magnet. The magnet behaves as a solenoid: this is what makes it work.

Picture B: fictitious magnetic surface charges

An alternative bookkeeping is to treat the divergence of **M** as a magnetic *surface charge density*,

$$\sigma_m = \hat{n} \cdot \vec{M},$$

negative on the south face (where the outward normal opposes **M**) and positive on the north face. These charges are fictitious — they do not correspond to real magnetic monopoles — but the fields they produce outside the magnet are correct, and the picture maps directly onto electrostatics.

The image-charge analogy

Now we exploit the analogy. A point charge $+q$ a distance d above an infinite grounded conductor at $V = 0$ produces the same field in the upper half-space as $+q$ together with an image charge $-q$ placed a distance d below the surface. The conductor is replaced by an equivalent free-space configuration in which the boundary condition is automatically satisfied.

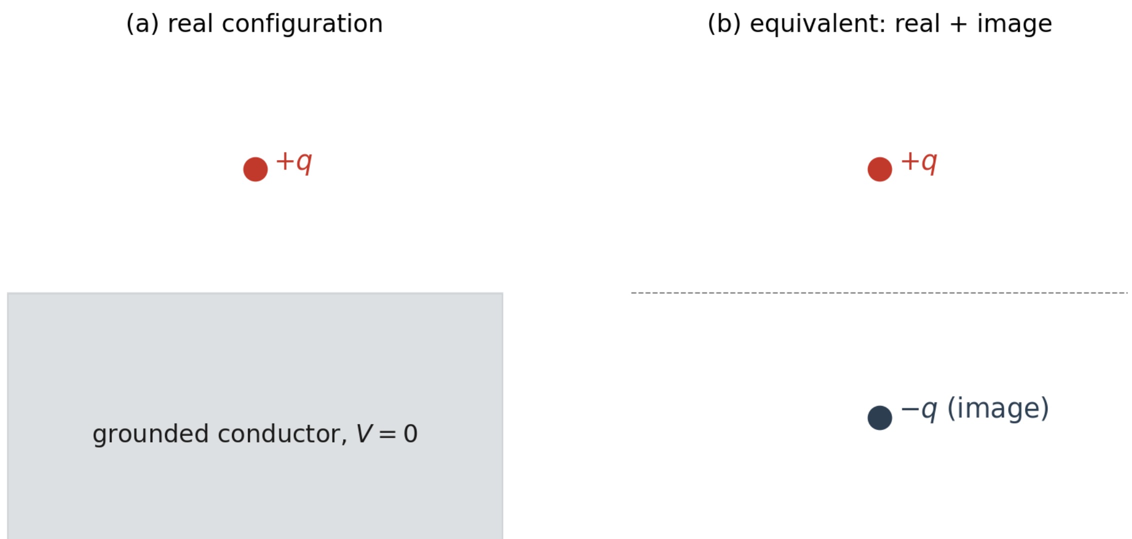

Figure 9. Left: a real charge +q above a grounded conductor. Right: the equivalent system in which the conductor is replaced by an image charge −q. The two systems produce the same field in the region above the original conductor.

The same trick works here. The Fe wall is a soft ferromagnet with a very large χ_f — in the idealized limit it acts like a "magnetic conductor", with the analogue of $V = 0$ being the requirement that **H** be normal to the surface. We replace the wall with an image bar magnet on the far side, oriented to make the field of the real magnet attract its own image. The N face of the real magnet pairs with an S image face just inside the wall, and the two faces attract.

Lifting force from the parallel-plate analogy

Pulling the magnet away from the wall is equivalent to pulling apart the two plates of a charged capacitor with surface charge density $\sigma \sim q/A$. The standard force-per-unit-area on a capacitor plate due to the field of its mate is $\sigma^2/(2\varepsilon_0)$; the magnetic analogue, in Gaussian-style cleanup, gives a holding force of order

$$F \sim \frac{\mu_0 M^2 A}{2},$$

with A the cross-sectional area of the bar at the contact face. Plugging in $M \sim 10^6$ A/m, $\mu_0 = 4\pi \times 10^{-7}$ T·m/A, and a contact area of (say) 1 $\text{cm}^2 = 10^{-4}$ m^2, gives an attractive force on the order of tens of newtons — consistent with everyday experience holding a fridge magnet against an iron surface.

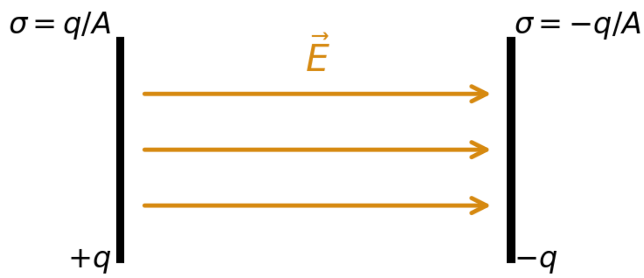

Figure 10. The force-on-plate analogy. The two charged plates feel an attractive force per unit area $\sigma^2/(2\varepsilon_0)$ from each other's fields. The magnetic version is what holds the magnet to the wall.

Big-picture takeaway from this problem: a hard ferromagnet on a soft ferromagnet wall is dual to a point charge above a grounded conductor. The image-charge trick that we built up in electrostatics carries over almost verbatim, because the boundary condition ***H*** *⊥ surface is the magnetic analogue of* ***E*** *⊥ surface (or equivalently V = constant). All the geometric reasoning is reusable; only the constants change.*