Fourier Series

Expand function $n(x)$ in sines + cosines.

Alternatively, could expand in polynomials

$$n(x) = A_0 + A_1 x + A_2 x^2 + \ldots$$

Taylor expansion

Hermite polynomials

Legendre Polynomials $P_\ell(\cos\theta)$

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expansion functions must "span the space" = complete

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also need orthonormality and analog of dot product.

Analog with vectors

$$\vec{n} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = \sum_{i=1}^{3} A_i \hat{e}_i$$

$\hat{\imath}$, $\hat{e}_x$

$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ = basis vectors

$\{A_1, A_2, A_3\}$ = coefficients

How do we solve for A_i?

$\hat{e}_1 \cdot \hat{e}_1 = 1$
$\hat{e}_2 \cdot \hat{e}_2 = 1$
$\hat{e}_1 \cdot \hat{e}_2 = 0$

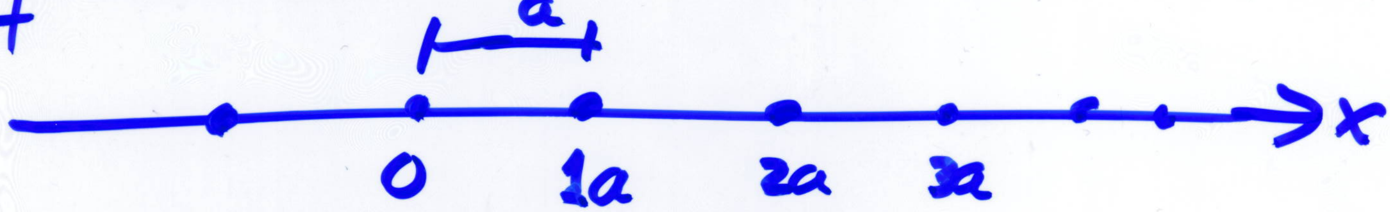
$$\hat{e}_j \cdot \vec{n} = \sum_i A_i \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\delta_{ij}}$$

$$= A_j$$

Now with functions

$$n(x) = \frac{A_0}{2} + \sum_{p=1}^{\infty} \left[A_p \cos(k_p x) + B_p \sin(k_p x) \right]$$

1-dimension crystal lattice



a = primitive cell

Direct lattice vector

$$\vec{R} = p\vec{a}$$

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k_p is a reciprocal lattice vector

$$k_p = \frac{2\pi p}{a}$$

Functional analog of dot product

Basis functions $\{1, \underbrace{\cos(k_p x), \sin(k_p x)}_{p \geq 1}\}$

like $\{\hat{e}_i\}$

$$\vec{v} \cdot \vec{w} = \langle v(x) | w(x) \rangle$$

$$\equiv \int_{x=x_0}^{x_0+a} v(x)\, w(x)\, dx$$

$$\langle 1 | 1 \rangle = \frac{2}{a} \int_{x=0}^{a} 1 \cdot 1 \cdot dx = 2$$

$\hat{e}_1 \cdot \hat{e}_1 = 1$

$$\langle 1 | \cos(k_p x) \rangle = \frac{2}{a} \int_{x=0}^{a} 1 \cdot \cos\left(\frac{2\pi p x}{a}\right) dx = 0$$

$\hat{e}_1 \cdot \hat{e}_2 = 0$

$$\langle 1 | \sin(k_p x) \rangle = 0$$

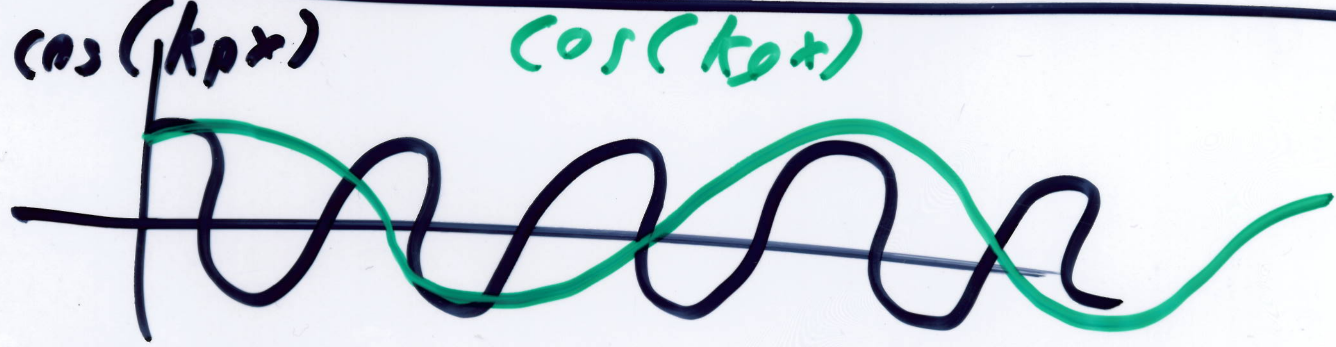
$$\langle \cos(k_1 x) | \cos(k_1 x) \rangle =$$

$$= \frac{2}{a} \int_{x=0}^{a} \cos^2\left(\frac{2\pi x}{a}\right) dx = 1$$

$$\langle \cos(k_p x) | \cos(k_\ell x) \rangle$$

$$= \frac{2}{a}\int_{x=0}^{a} \cos\left(\frac{2\pi p x}{a}\right)\cos\left(\frac{2\pi \ell x}{a}\right)dx = \delta_{p\ell}$$

$$\langle \sin(k_p x) | \sin(k_\ell x) \rangle = \delta_{p\ell}$$



$$\langle \cos(k_p x) | \sin(k_\ell x) \rangle = 0$$

$$n(x) = \frac{A_0}{2}1 + \sum_{p=1}^{\infty}\left[A_p \cos(k_p x) + B_p \sin(k_p x)\right]$$

$$\boxed{\ell > 0}$$

$$\langle n(x) | \cos(k_\ell x)\rangle$$

$$= \frac{A_0}{2}\cancelto{0}{\langle 1 | \cos(k_\ell x)\rangle}$$

$$+ \sum_{p=1}^{\infty} A_p \overbrace{\langle \cos(k_p x) | \cos(k_\ell x)\rangle}^{\delta_{p\ell}}$$

$$+ \sum_{p=1}^{\infty} B_p \cancelto{0}{\langle \sin(k_p x) | \cos(k_\ell x)\rangle} = A_\ell$$

$$A_\ell = \frac{2}{a}\int_{x=0}^{a} n(x)\cos\left(\frac{2\pi\ell}{a}x\right)dx$$

$$B_\ell = \ldots \sin(\quad) \ldots$$