# Dispersion Relation $\omega(k)$

e.g. Light in vacuum

D.R. $\omega = c\,k$

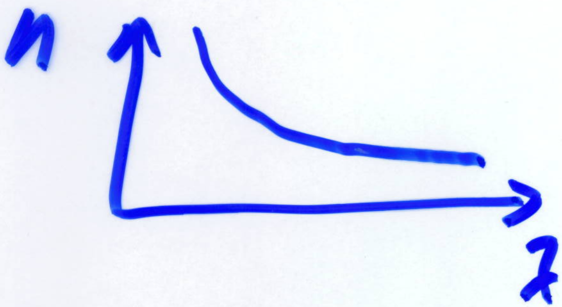
$\uparrow$ speed of light

e.g. Light in a material with index of refraction $n(\omega)$

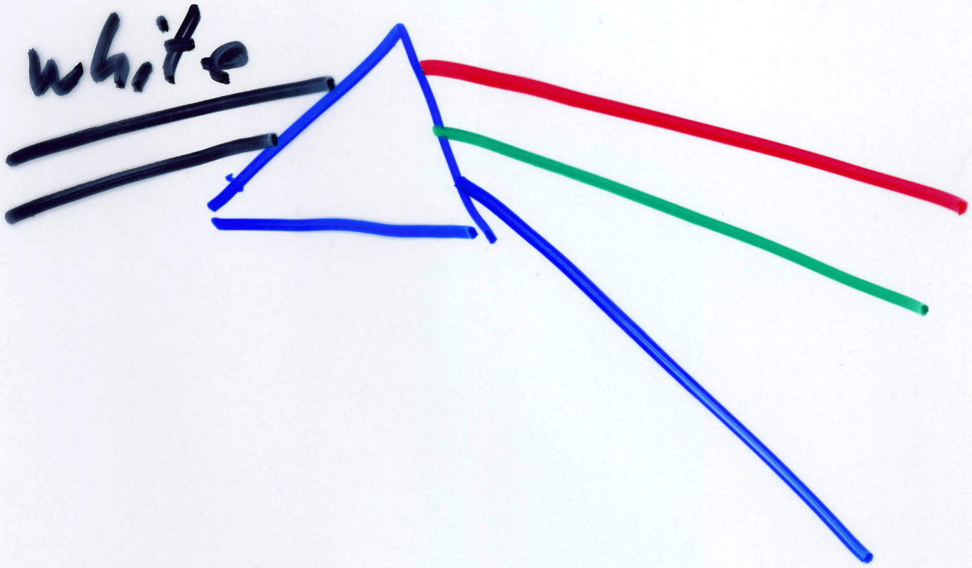
$n = 1$ vacuum

$n \geq 1$

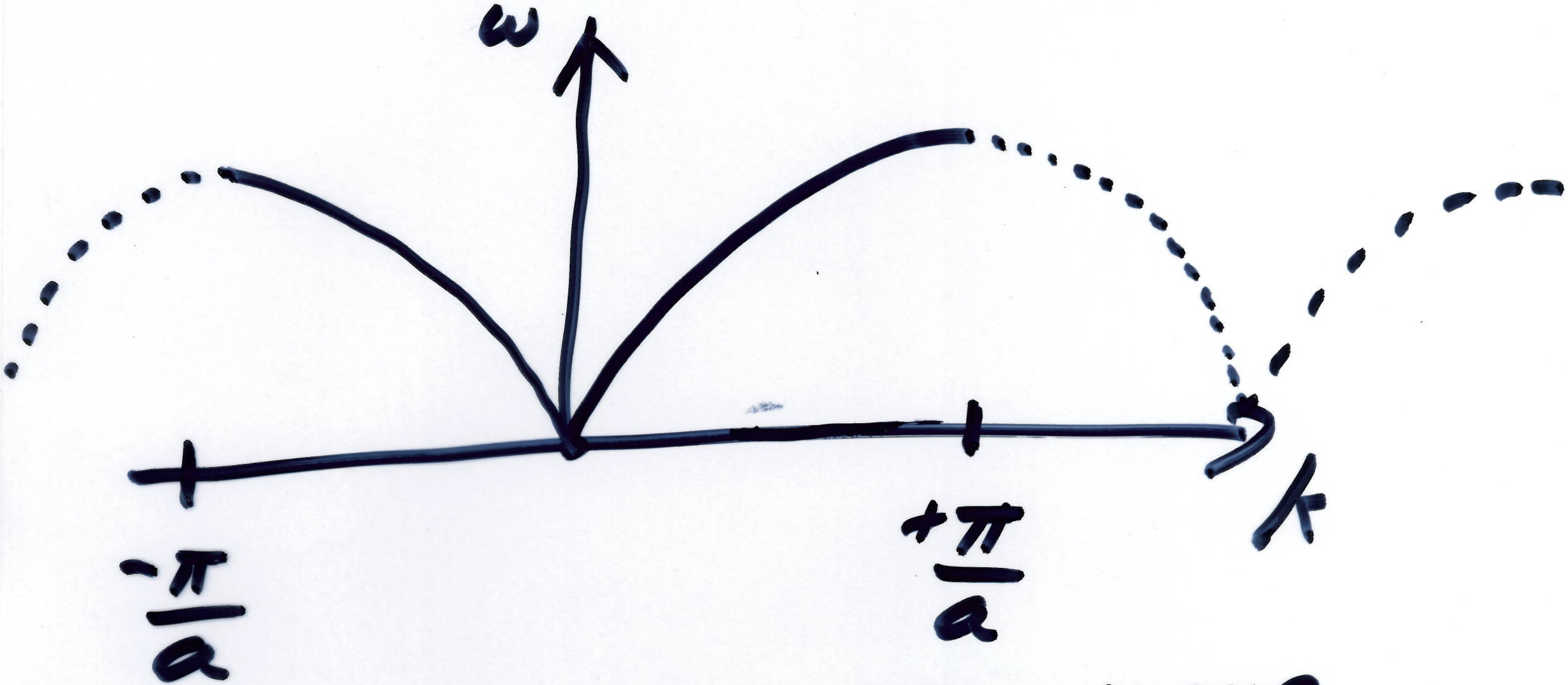
D.R. $\omega = \dfrac{c}{n(\omega)} k$







D.R. $\omega = 2\sqrt{\frac{C}{M}}\left|\sin\left(\frac{ka}{2}\right)\right|$




Suppose we send a __wave packet__ into the crystal

Two velocities

phase velocity $v_\varphi = \frac{\omega}{k}$

group velocity $v_g = \frac{d\omega}{dk}$

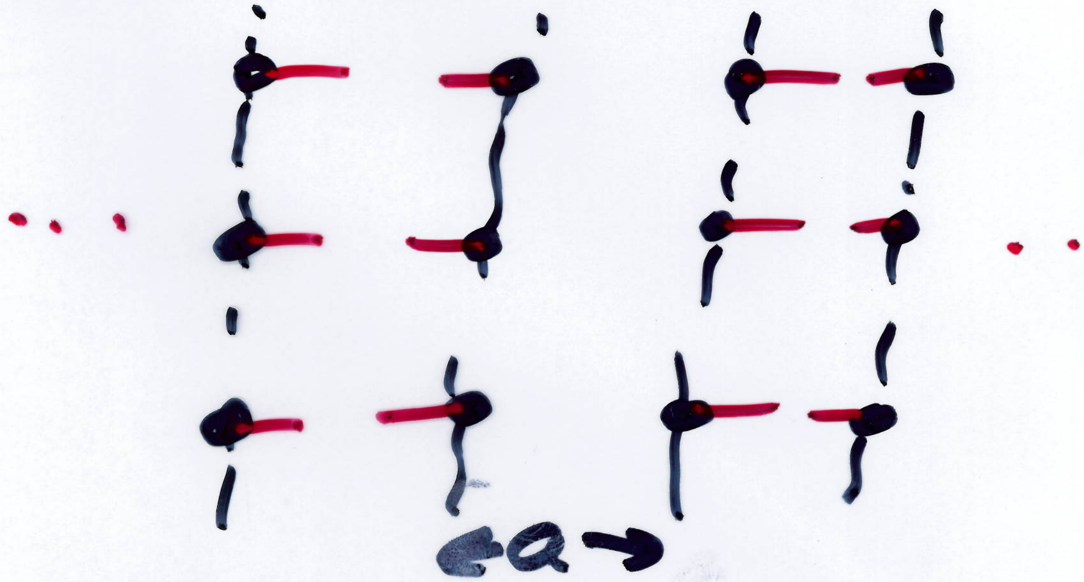
$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left[2\sqrt{\frac{C}{M}}\sin\left(\frac{ka}{2}\right)\right]$$

$$= a\sqrt{\frac{C}{M}}\cos\left(\frac{ka}{2}\right)$$


$v_g$ is tangent slope

first Brillouin zone

At the zone boundaries $k = \pm\frac{\pi}{a}$




standing wave
$v_g = 0$

$$\frac{u_{s+1}}{u_s} = \frac{A e^{ika(s+1)}}{A e^{ikas}} = e^{ika}$$

At zone boundary $ka = \pm \pi$

$$\frac{u_{s+1}}{u_s} = e^{\pm i\pi} = -1$$

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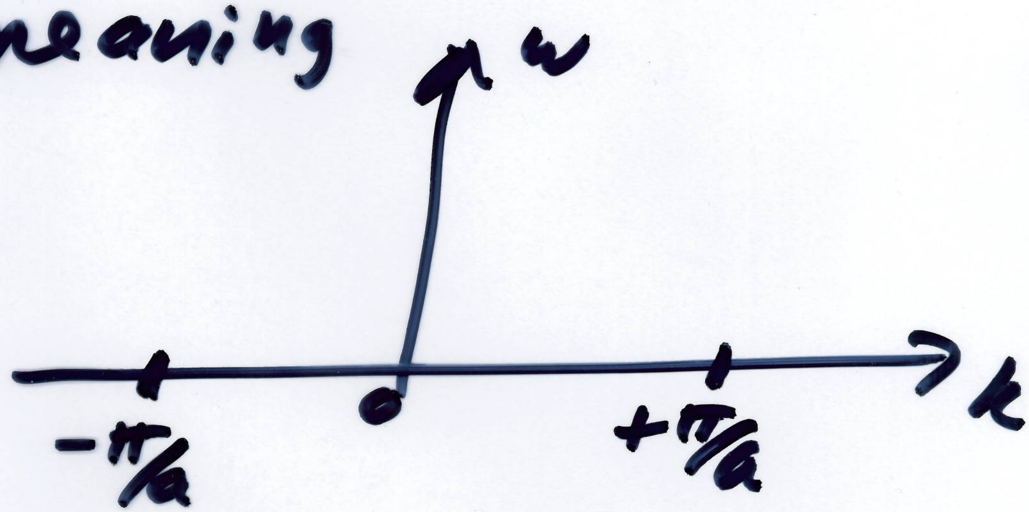
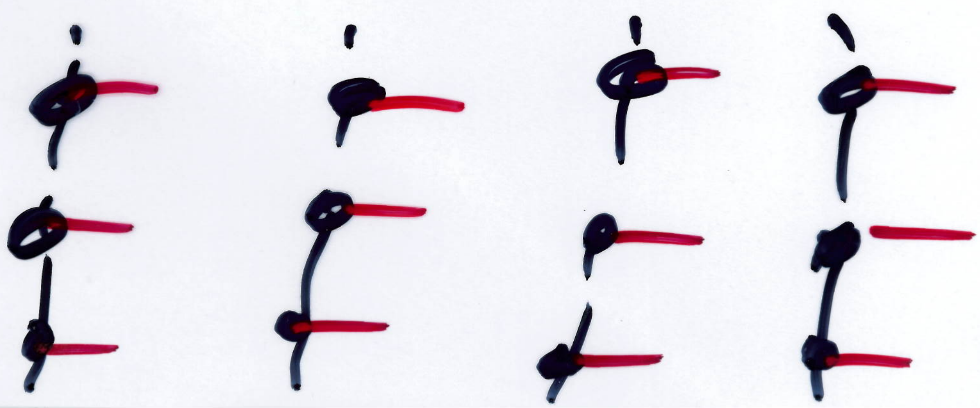
Only wavevectors $k$ in the first Brillouin have physical meaning

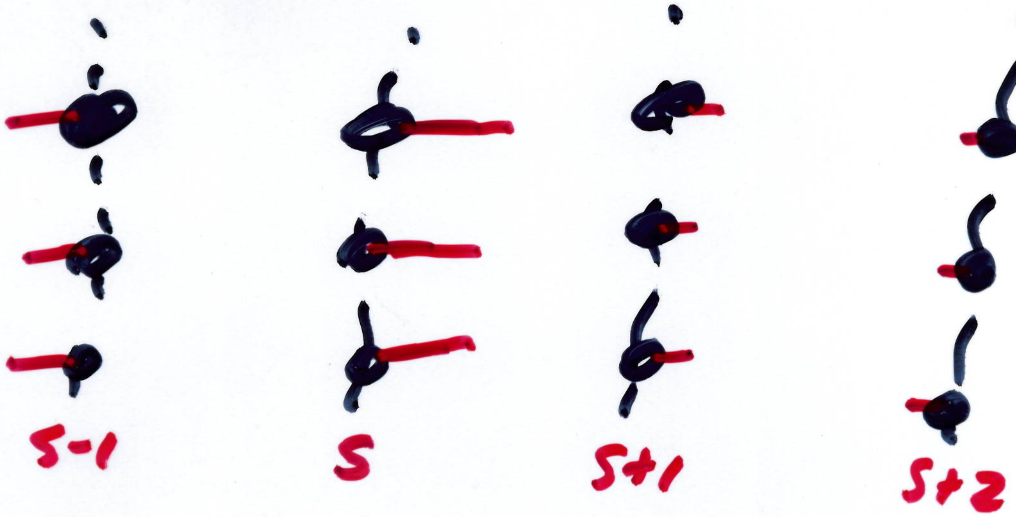
$$k = \frac{2\pi}{\lambda}$$



$\lambda = a$

not allowed




$$\lambda = \frac{a\,17}{\sqrt{3}}$$

For $k$'s outside the Brillouin zone $|k| > \pi/a$

$$u_s = A e^{ikas} = A e^{ias(k_{in} + 2\pi n)}$$

$$= A e^{ias(k_{in})}$$

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Long wavelength limit
(small $k$ limit)

D.R. $\omega^2 = \frac{2C}{M}\left[1 - \cos(ka)\right]$

$\lambda \gg a$, $ka \ll 1$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$$

$$\omega^2 = \frac{2C}{m}\left[1 - \left(1 - \frac{k^2a^2}{2}\right)\right] + \dots$$

$$\omega^2 = \frac{2C}{m}\frac{k^2a^2}{2}$$

$$\omega = a\sqrt{\frac{C}{m}}\,k$$

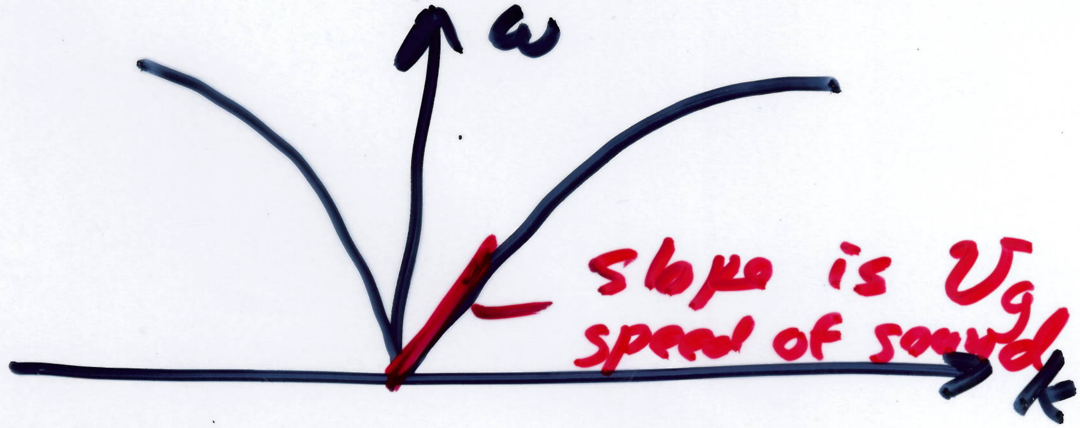
$\omega \propto k$

no dispersion

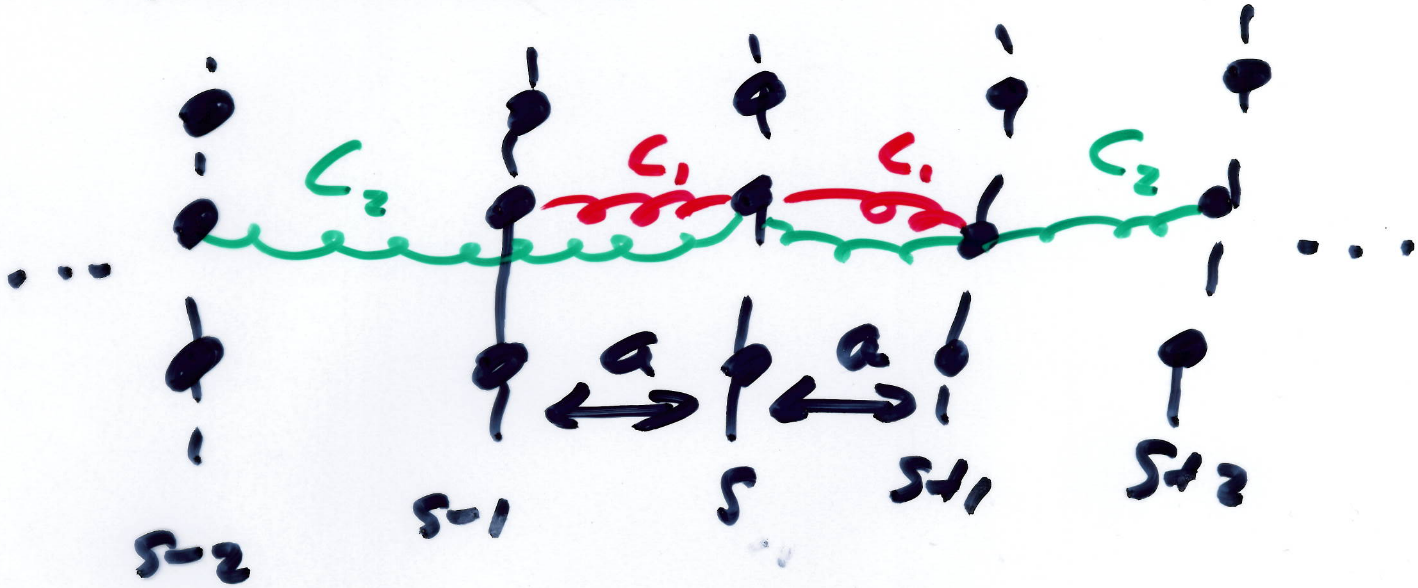
speed of sound (long $\lambda$'s)

no dispersion : $v_g = v_\varphi$

$$v_g = \frac{\omega}{k} = a\sqrt{\frac{C}{m}} = v_\varphi = \frac{d\omega}{dk}$$



Longitudinal waves
more than nearest neighbor interactions

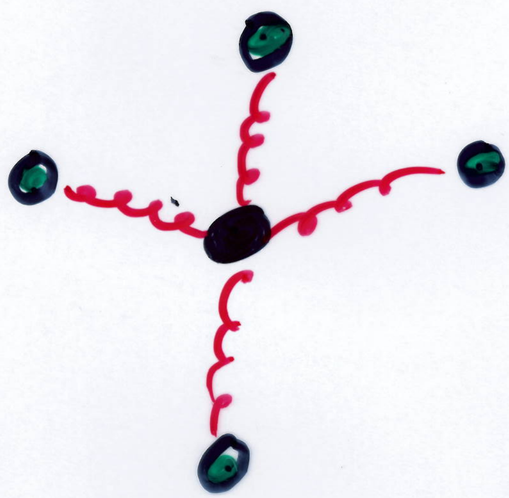


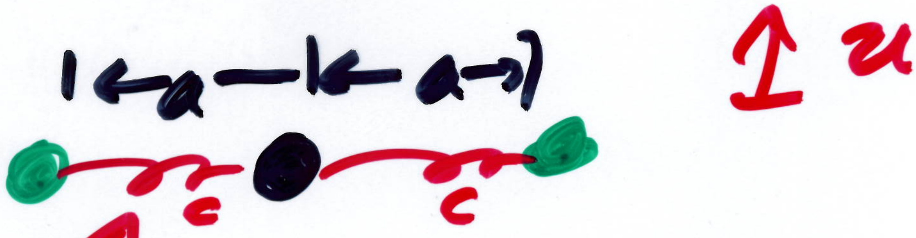

$$\omega^2 = \frac{2C}{M}\left[1-\cos(ka)\right]$$

$$\omega^2 = \frac{2}{M}\sum_{p=1}^{?} C_p\left[1-\cos(kpa)\right]$$

# Transverse waves


No contribution from nearest neighbor interactions






If these springs are under tension (rest length < a) there is a restoring force proportional to displacement.

$F \propto C_{eff}\, u$

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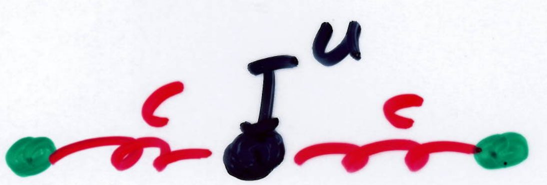
Our problem




Rest length of springs = a

$F \propto u^3$


new length is $\sqrt{a^2+u^2}$

amount spring stretched is

$\sqrt{a^2+u^2} - a$ $\quad u \ll a$

$$= a\sqrt{1+\frac{u^2}{a^2}} - a$$

$$= a\left(1+\frac{u^2}{a^2}\right)^{1/2} - a$$

$$= a\left[1 + \frac{u^2}{2a^2} + \cdots\right] - a$$

$$= \frac{u^2}{a}$$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$$\text{Force} = F = C\frac{u^2}{a}$$

$$F_y = F\sin\theta \approx F\tan\theta \approx F\frac{u}{a}$$

$$2F_y = F_{tot} = C\frac{u^3}{a^2} \propto u^3$$