$$-\omega^2 A m_1 = cB[1+e^{-ika}] - 2cA$$

$$-\omega^2 B m_2 = cA[e^{+ika}+1] - 2cB$$

$$(2c-\omega^2 m_1)A - c[1+e^{-ika}]B = 0$$

$$-c[e^{+ika}+1]A + (2c-\omega^2 m_2)B = 0$$

$$\begin{bmatrix} 2c-\omega^2 m_1 & -c(1+e^{-ika}) \\ -c(e^{+ika}+1) & 2c-\omega^2 m_2 \end{bmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\equiv \quad \underline{\underline{M}}\cdot\vec{v} = \vec{0}$$

If $\underline{\underline{M}}$ had an inverse: $\underline{\underline{M}}^{-1}\underline{\underline{M}}\cdot\vec{v} = \underline{\underline{M}}^{-1}\cdot\vec{0}$

$$\Rightarrow \mathbb{I}\cdot\vec{v} = \vec{v} = \vec{0} \Rightarrow A=0,\ B=0$$

Non-trivial solutions

$\nexists\ \underline{\underline{M}}^{-1}$ $\qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\underline{\underline{M}}^{-1} = \frac{1}{\det(\underline{\underline{M}})} \operatorname{adj}(\underline{\underline{M}})$$

If $\det(\underline{\underline{M}}) = 0$ then there is no $\underline{\underline{M}}^{-1}$

$$\det(\underline{\underline{M}}) = (2C - \omega^2 m_1)(2C - \omega^2 m_2) - C^2(1 + e^{ika})(1 + e^{-ika}) = 0$$

$$\Rightarrow 4C^2 - 2C\omega^2(m_1 + m_2) + \omega^4 m_1 m_2 - C^2(2 + e^{ika} + e^{-ika})$$

$$\Rightarrow m_1 m_2 (\omega^2)^2 - 2C(m_1 + m_2)\omega^2 + 2C^2[1 - \cos(ka)]$$

Quadratic equation in ω^2

$$a(\omega^2)^2 + b\omega^2 + c = 0$$

$$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Reduced Mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\mu < m_1, m_2$$

$$\omega^2 = C\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm C\sqrt{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{4}{m_1 m_2}\sin^2\left(\frac{ka}{2}\right)}$$

$$\omega^2 = \frac{C}{\mu} \pm C\sqrt{\frac{1}{\mu^2} - \frac{4}{m_1 m_2}\sin^2\left(\frac{ka}{2}\right)}$$

Long wavelength limit $ka \ll 1$

$$\cos(ka) = 1 - \frac{1}{2}k^2a^2 + O(ka)^4 \ldots$$

$$\det(\underline{\underline{M}}) = \omega^4 m_1 m_2 - 2C(m_1+m_2)\omega^2 + C^2k^2a^2 = 0$$

$$\omega^2 = \frac{2C(m_1+m_2) \pm \sqrt{4C^2(m_1+m_2)^2 - 4m_1m_2C^2k^2a^2}}{2m_1 m_2}$$

$$\oplus\ \omega^2 = \frac{2C(m_1+m_2) + 2C(m_1+m_2)}{2m_1 m_2} = 2C\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

$$= \frac{2C}{\mu} \text{ optical}$$

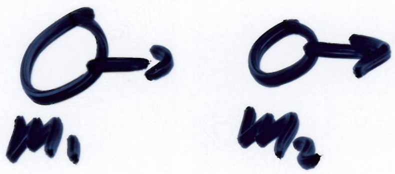
$$\ominus\ \omega^2 = \frac{2C(m_1+m_2) - 2C(m_1+m_2)\sqrt{1 - \frac{k^2a^2m_1m_2}{(m_1+m_2)^2}}}{2m_1 m_2}$$

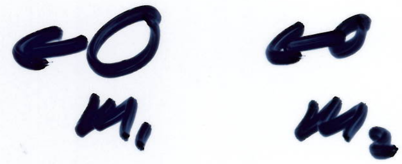
Binomial expansion

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}} \simeq 1 - \frac{x}{2} + O(x^2)\ldots$$

$$\omega^2 = \frac{2C(m_1+m_2)k^2a^2m_1m_2}{2(m_1+m_2)^2} = \frac{Ck^2a^2}{2(m_1+m_2)} \text{ acoustic}$$

acoustic mode ; M_{TOTAL}





Center of mass moves

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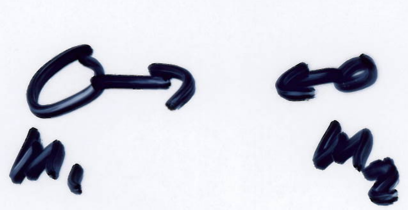
optical mode ; μ - reduced mass



Center of mass stays fixed

At the Brillouin zone Boundary

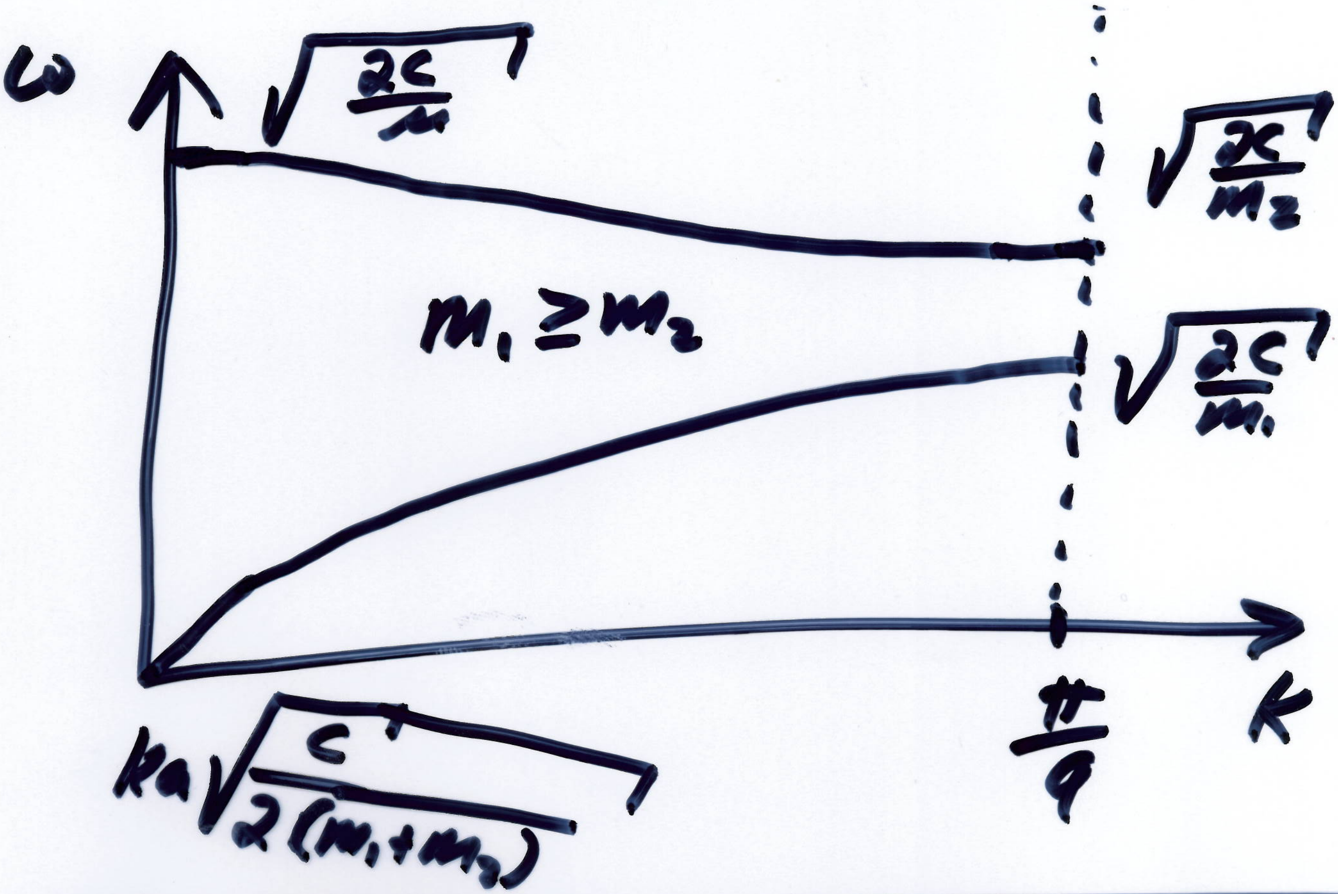
$$ka = \pm\pi$$

$$\cos(ka) \Longrightarrow \cos(\pm\pi) = -1$$

$$\det(\underline{\underline{M}}) = \omega^4 m_1 m_2 - 2C(m_1+m_2)\omega^2 + 4C^2$$

$$\Rightarrow \omega^2 = \frac{2C(m_1+m_2) \pm \sqrt{4C^2(m_1+m_2)^2 - 16C^2 m_1 m_2}}{2m_1 m_2}$$

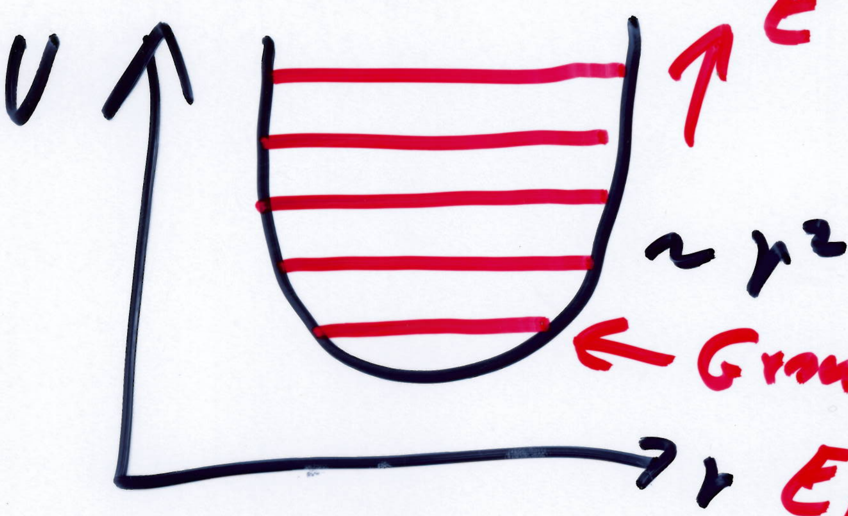
⊕ $\omega^2 = \frac{2c}{m_2}$ ⊖ $\omega^2 = \frac{2c}{m_1}$



Why "optical" If crystal is ionic

$\underline{a}$ lattice spacing ~ 2 Å

Visible light 4000-7000 Å

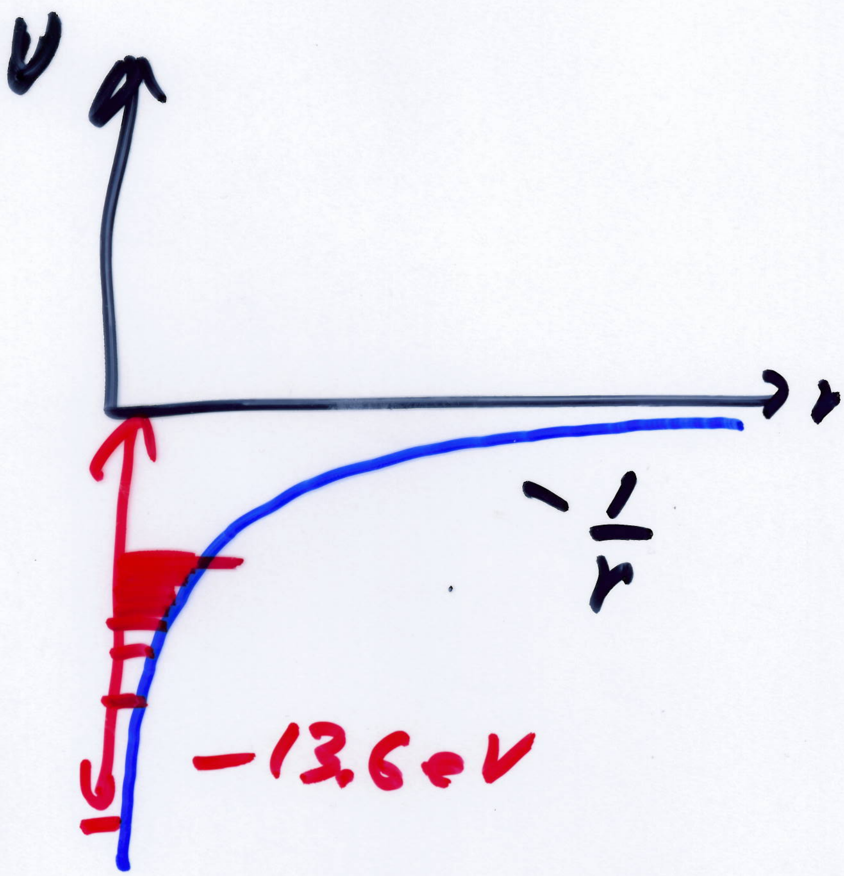


Simple harmonic oscillator

$E_1 = (\frac{3}{2})\hbar\omega$

Ground state $E_0 = \frac{1}{2}\hbar\omega$

$E_n = (n+\frac{1}{2})\hbar\omega$



Hydrogen