■ Equal Masses

In[70]:=
```
r = {c → 1, m1 → 2.2, m2 → 2.2};
```

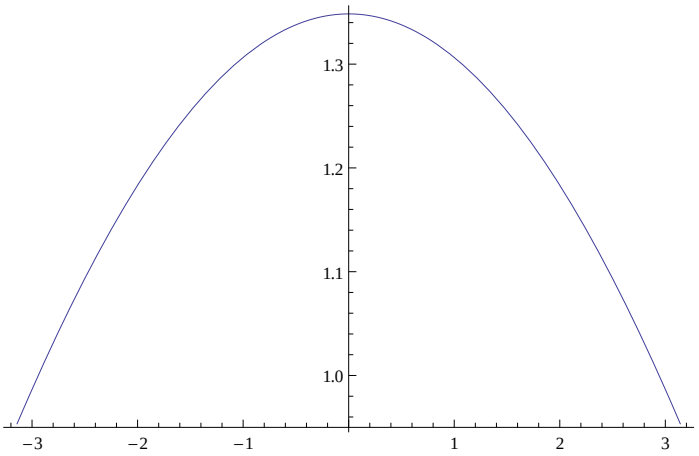
In[71]:=
```
μ = m1 m2 / (m1 + m2);
```

In[72]:=
```
ωo[ka_] = Sqrt[c / μ + c Sqrt[1 / μ^2 - 4 / (m1 m2) Sin[ka / 2]^2]]
```

Out[72]=
$$\sqrt{\frac{c\ (m1+m2)}{m1\ m2}+c\sqrt{\frac{(m1+m2)^2}{m1^2\ m2^2}-\frac{4\ \text{Sin}\left[\frac{ka}{2}\right]^2}{m1\ m2}}}$$

In[73]:=
```
p1 = Plot[ωo[ka] /. r, {ka, -π, π}]
```

Out[73]=

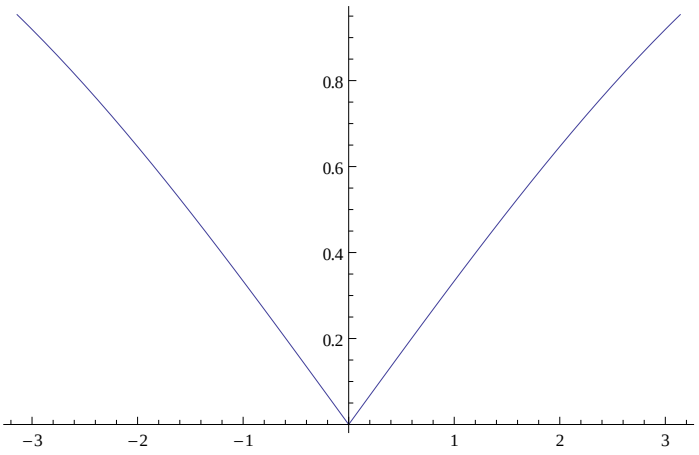


In[74]:=
```
ωa[ka_] = Sqrt[c / μ - c Sqrt[1 / μ^2 - 4 / (m1 m2) Sin[ka / 2]^2]]
```

Out[74]=
$$\sqrt{\frac{c\ (m1+m2)}{m1\ m2}-c\sqrt{\frac{(m1+m2)^2}{m1^2\ m2^2}-\frac{4\ \text{Sin}\left[\frac{ka}{2}\right]^2}{m1\ m2}}}$$

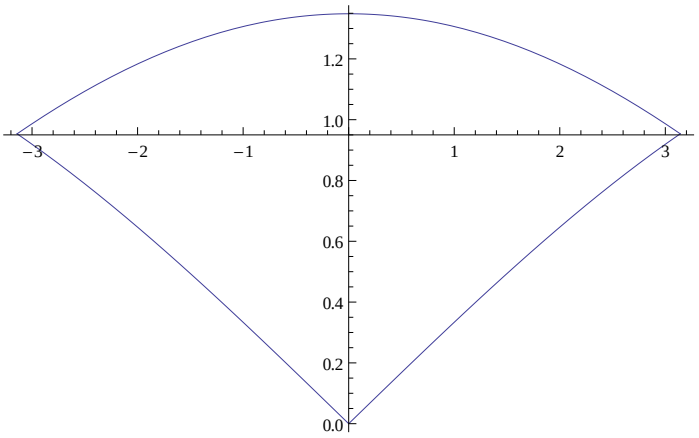
In[75]:=
```
p2 = Plot[ωa[ka] /. r, {ka, -π, π}]
```

Out[75]=



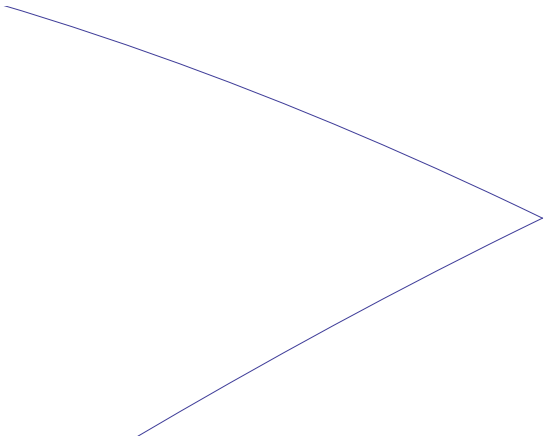
In[76]:= **Show[p1, p2, PlotRange → All]**

Out[76]=



In[77]:= **Show[p1, p2, PlotRange → {{1.5, π}, {0.7, 1.2}}, Axes → False]**

Out[77]=



In[78]:= **dωa[ka_] = D[Sqrt[c/μ − c Sqrt[1/μ^2 − 4/(m1 m2) Sin[ka/2]^2]], ka]**

Out[78]=

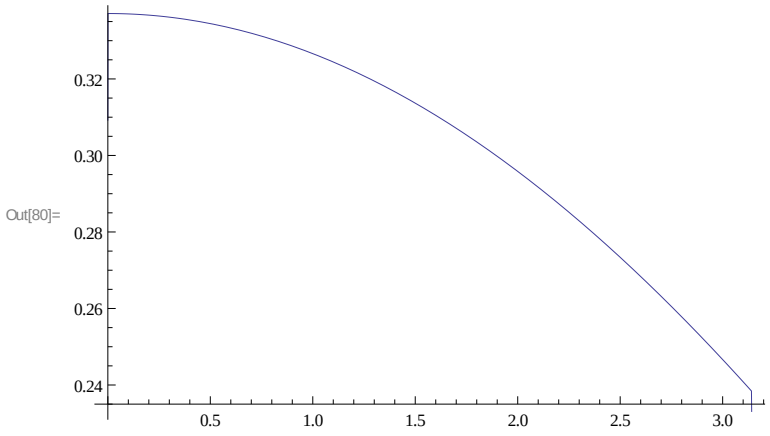
$$\left(c \cos\left[\frac{ka}{2}\right] \sin\left[\frac{ka}{2}\right]\right) \Bigg/ \left(m1\, m2 \sqrt{\frac{(m1+m2)^2}{m1^2\, m2^2} - \frac{4 \sin\left[\frac{ka}{2}\right]^2}{m1\, m2}} \sqrt{\frac{c\,(m1+m2)}{m1\, m2} - c \sqrt{\frac{(m1+m2)^2}{m1^2\, m2^2} - \frac{4 \sin\left[\frac{ka}{2}\right]^2}{m1\, m2}}}\right)$$

In[79]:= **Limit[dωa[ka] /. r, ka → π]**

Out[79]= −0.238366

In[80]:=
```
Plot[dωa[ka] /. r, {ka, 0, π}]
```

Out[80]=



■ Unequal Masses

In[81]:=
```
r = {c → 1, m1 → 2.3, m2 → 2.2};
```

In[82]:=
```
μ = m1 m2 / (m1 + m2);
```

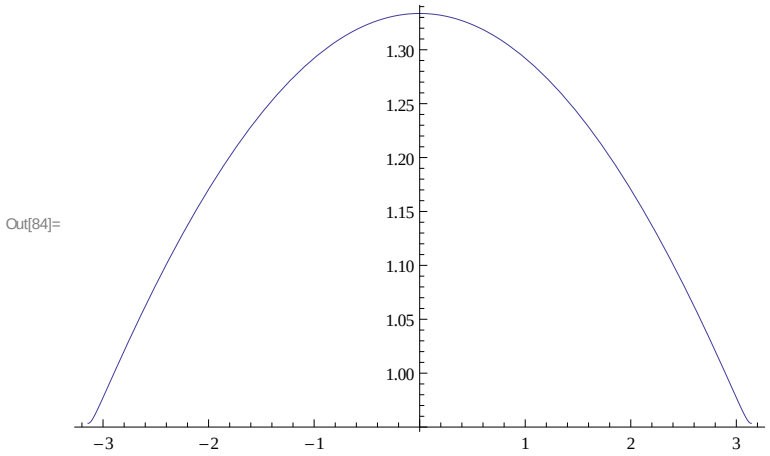
In[83]:=
```
ωo[ka_] = Sqrt[c / μ + c Sqrt[1 / μ^2 - 4 / (m1 m2) Sin[ka / 2]^2]]
```

Out[83]=

$$\sqrt{\frac{c\ (m1+m2)}{m1\ m2}+c\sqrt{\frac{(m1+m2)^2}{m1^2\ m2^2}-\frac{4\ \text{Sin}\left[\frac{ka}{2}\right]^2}{m1\ m2}}}$$

In[84]:=
```
p1 = Plot[ωo[ka] /. r, {ka, -π, π}]
```

Out[84]=



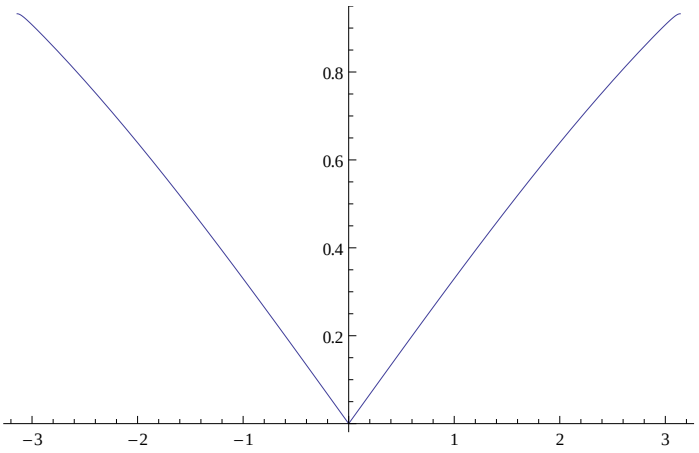
In[85]:=
```
ωa[ka_] = Sqrt[c / μ - c Sqrt[1 / μ^2 - 4 / (m1 m2) Sin[ka / 2]^2]]
```

Out[85]=

$$\sqrt{\frac{c\ (m1+m2)}{m1\ m2}-c\sqrt{\frac{(m1+m2)^2}{m1^2\ m2^2}-\frac{4\ \text{Sin}\left[\frac{ka}{2}\right]^2}{m1\ m2}}}$$

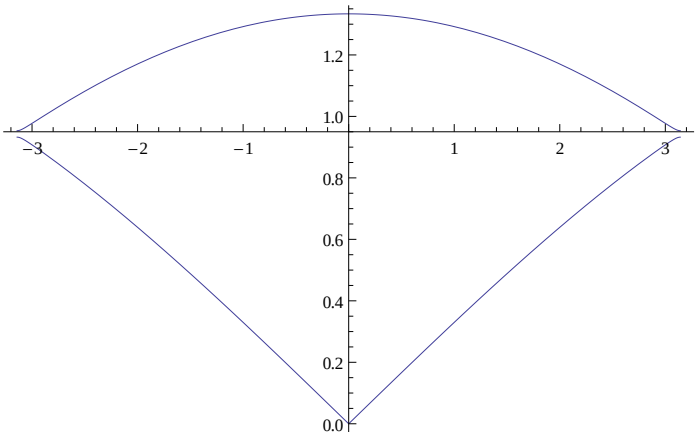
In[86]:=
```
p2 = Plot[ωa[ka] /. r, {ka, -π, π}]
```

Out[86]=



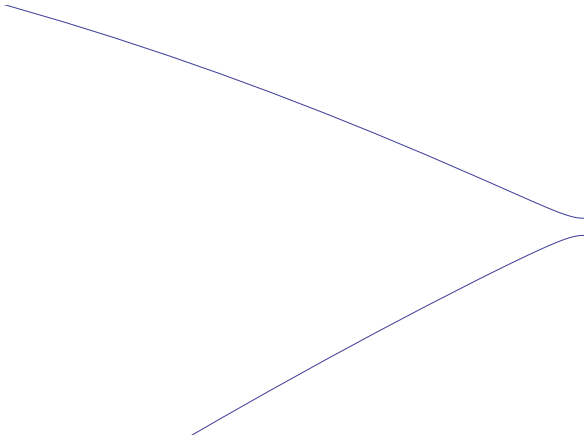
In[87]:=
```
Show[p1, p2, PlotRange → All]
```

Out[87]=



In[88]:=
```
Show[p1, p2, PlotRange → {{1.5, π}, {0.7, 1.2}}, Axes → False]
```

Out[88]=



In[89]:=
```
dωa[ka_] = D[Sqrt[c/μ - c Sqrt[1/μ^2 - 4/(m1 m2) Sin[ka/2]^2]], ka]
```

Out[89]=

$$\left(c \cos\left[\frac{ka}{2}\right] \sin\left[\frac{ka}{2}\right]\right) \Bigg/ \left(m1\, m2 \sqrt{\frac{(m1+m2)^2}{m1^2\, m2^2} - \frac{4 \sin\left[\frac{ka}{2}\right]^2}{m1\, m2}} \sqrt{\frac{c\,(m1+m2)}{m1\, m2} - c \sqrt{\frac{(m1+m2)^2}{m1^2\, m2^2} - \frac{4 \sin\left[\frac{ka}{2}\right]^2}{m1\, m2}}}\right)$$

In[90]:=
```
Limit[dωa[ka] /. r, ka → π]
```

Out[90]=
```
0.
```

In[91]:=
```
Plot[dωa[ka] /. r, {ka, 0, π}]
```

Out[91]=