Georg Cantor (GAY-org)

Diagonalization Proof

"There are more real $\overset{(0,1)}{\text{numbers}}$ than integers."

$r_1 = 0.794222537\ldots$

$r_2 = 0.821214768\ldots$

$r_3 = 0.111111111\ldots$

$r_4 = 0.00023450000\ldots$

$\vdots$

$r_{new} = 0.5190$

Aleph-null

The "number" of integers $= \aleph_0$

The "number" of reals $= \underset{\text{continuum}}{C} = \aleph_1$ (sometimes)

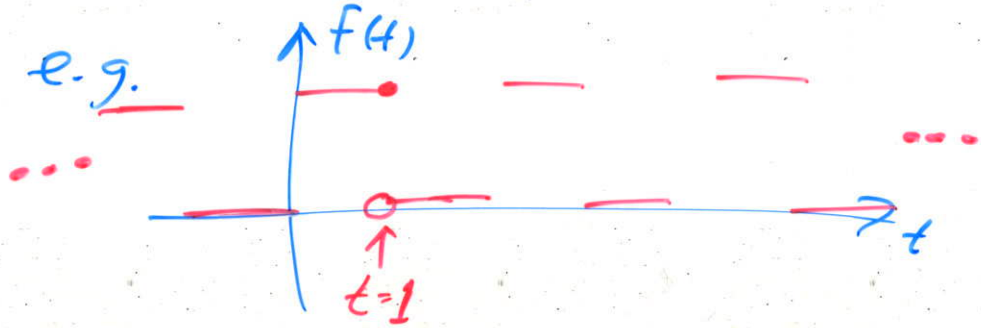
$\aleph_1 = 2^{\aleph_0}$

$\aleph_2 = 2^{\aleph_1}$

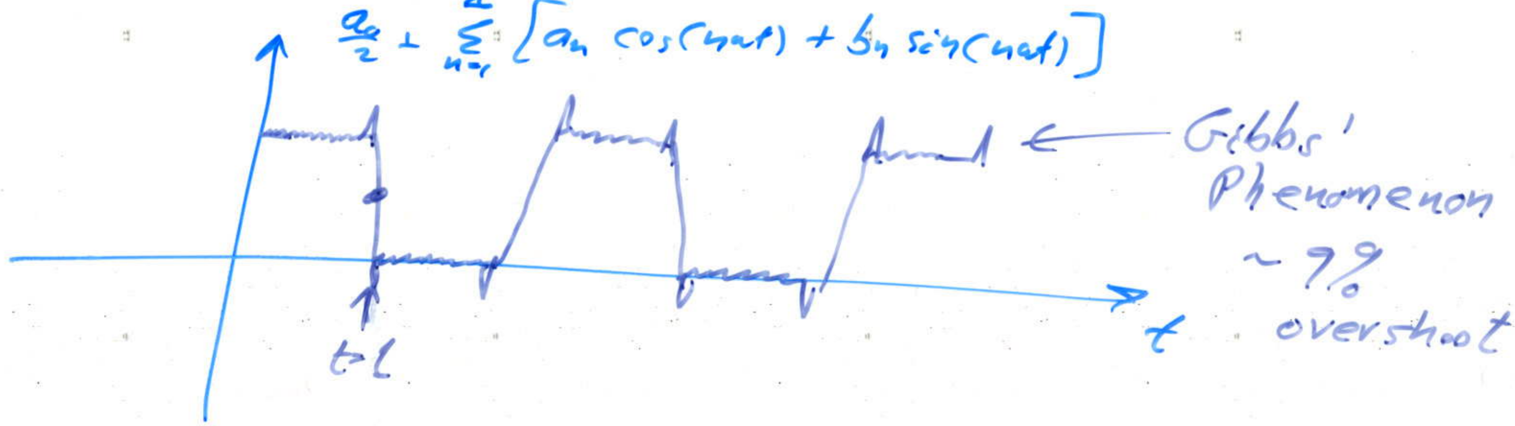
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

What if $f(t)$ is discontinuous? (Sine and cosine are continuous.)

e.g.



The Fourier series converges in the mean to $f(t)$, not pointwise convergence.



Sines and Cosines are complete.

Why sometimes are half of the Fourier coefficients equal to zero?

Ans. There is a symmetry about the point $t = \frac{T}{4}$.

Complex Fourier Series

$i = \sqrt{-1}$

Euler: $e^{i\theta} = \cos\theta + i\sin\theta$

① $$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t} \qquad \omega = \frac{2\pi}{T}$$

Find c_p: $$\left\langle e^{ip\omega t} \middle| f(t) \right\rangle \equiv \frac{1}{T}\int_{t'=0}^{T} e^{-ip\omega t'} f(t')\,dt'$$

complex conjugate

$(e^{ip\omega t})^*$

Completeness

Substitute c_n back into ①

$$f(t) = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{T}\int_{t'=0}^{T} e^{-in\omega t'} f(t')\,dt'\right] e^{in\omega t}$$

$$f(t) = \int_{t'=0}^{T} \underbrace{\left[\frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{in\omega(t-t')}\right]}_{\delta(t-t')} f(t')\,dt'$$

Dirac delta "function"

like Kronecker delta δ_{np}

Proof: $\displaystyle \frac{1}{T}\int_0^T e^{-ip\omega t}\, e^{in\omega t}\, dt = \delta_{np} = \langle e^{ip\omega t} | e^{in\omega t} \rangle$

$$= \frac{1}{T}\int_0^T \left[\cos(p\omega t) - i\sin(p\omega t)\right]\left[\cos(n\omega t) + i\sin(n\omega t)\right] dt$$

$$\left.\begin{aligned}
&= \frac{1}{T}\int_0^T \cos(p\omega t)\cos(n\omega t)\, dt = \frac{1}{2}\delta_{pn} \\
&+ \frac{1}{T}\int_0^T \sin(p\omega t)\sin(n\omega t)\, dt = \frac{1}{2}\delta_{pn} \\
&+ \frac{i}{T}\int_0^T \cos(p\omega t)\sin(n\omega t)\, dt = 0 \\
&- \frac{i}{T}\int_0^T \sin(p\omega t)\cos(n\omega t)\, dt = 0
\end{aligned}\right\}\ \delta_{np} \quad \blacksquare$$

Orthogonality / Normality

What if $f(t)$ is a real function? $f^*(t) = f(t)$

$$c_n = \langle e^{in\omega t} | f(t) \rangle = \frac{1}{T}\int_0^T e^{-in\omega t} f(t)\, dt$$

$$= \frac{1}{T}\int_0^T \cos(n\omega t) f(t)\, dt + \frac{-i}{T}\int_0^T \sin(n\omega t) f(t)\, dt$$

$$= \frac{1}{2} a_n \qquad - i\frac{1}{2} b_n \implies c_n = \frac{1}{2}(a_n - i b_n)$$

$$c_{-n} = \frac{1}{T}\int_0^T e^{+in\omega t} f(t)\, dt = c_n^* = \frac{1}{2}(a_n + i b_n)$$

$$c_0 = \frac{1}{T}\int_0^T e^0 f(t)\, dt = \frac{a_0}{2} = (f)_{avg}$$

Completeness

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t} \qquad c_n = \frac{1}{T}\int_0^T e^{-in\omega t'} f(t')\, dt'$$

substitute

$$f(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T}\int_{t'=0}^{T} e^{-in\omega t'} f(t')\, dt' \right] e^{in\omega t}$$

$$f(t) = \int_{t'=0}^{T} \underbrace{\left[\frac{1}{T} \sum_{n=-\infty}^{+\infty} e^{in\omega(t-t')} \right]}_{\delta(t-t')} f(t')\, dt'$$

What if $f(t)$ is <u>not</u> periodic and is defined on the entire t axis. $\longrightarrow$ Fourier Transform

Limits $T \to \infty$, $\omega \to d\omega$,

$$\sum_n \to \int_\omega \qquad n\omega \to \omega \qquad T = \frac{2\pi}{\omega}, \quad \frac{1}{T} = \frac{\omega}{2\pi} \to \frac{d\omega}{2\pi}$$

$$f(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\left[\frac{1}{T} \int_{t'=-\frac{T}{2}}^{+\frac{T}{2}} e^{-in\omega t'} f(t')\,dt' \right]}_{\text{In the limit, this is } \tilde{f}(\omega)} e^{in\omega t}$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{+\infty} e^{-i\omega t} f(t)\,dt$$

Fourier transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{+\infty} e^{i\omega t} \underset{\uparrow\ \text{"}C(\omega)\text{"}}{\tilde{f}(\omega)}\,d\omega$$

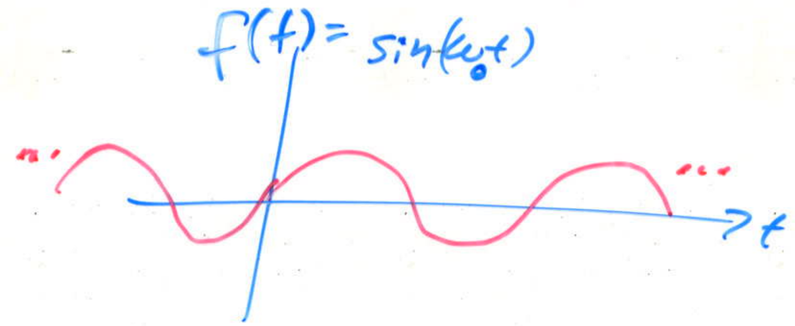
Inverse Fourier transform

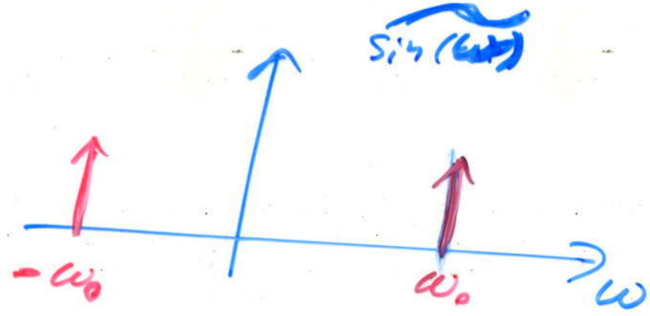
Why?

Time Domain | Frequency Domain

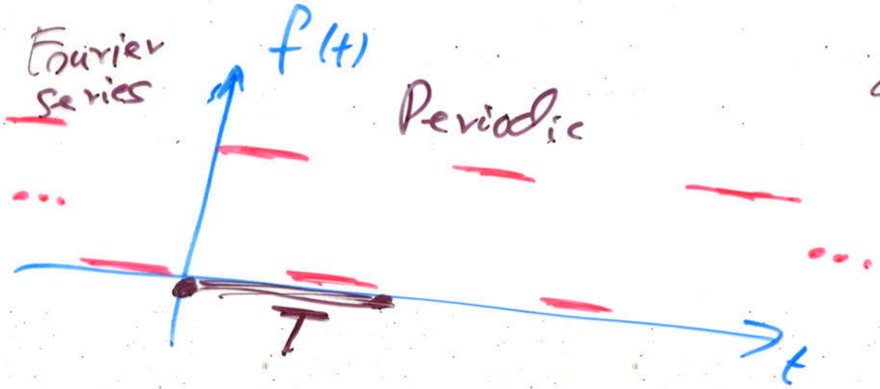
Given a differential equation – asked to solve for $f(t)$ $\xrightarrow{\text{Fourier transform}}$ Algebraic equation for $\tilde{f}(\omega)$ – solve for $\tilde{f}(\omega)$

$f(t)$ $\xleftarrow{\text{Inverse Fourier transform}}$

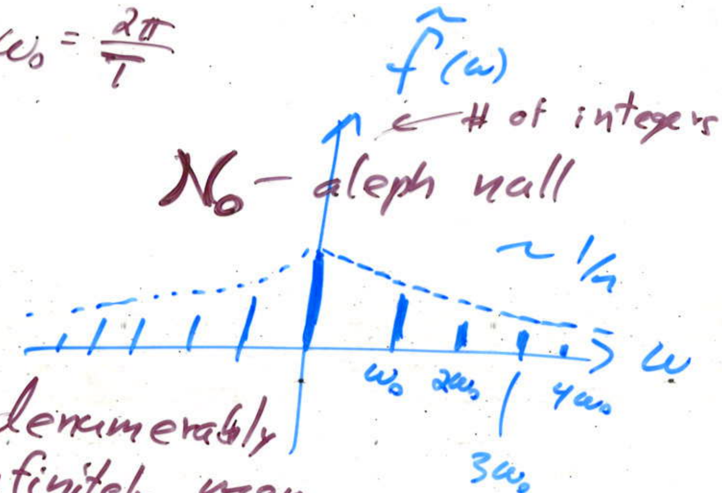




Fourier series



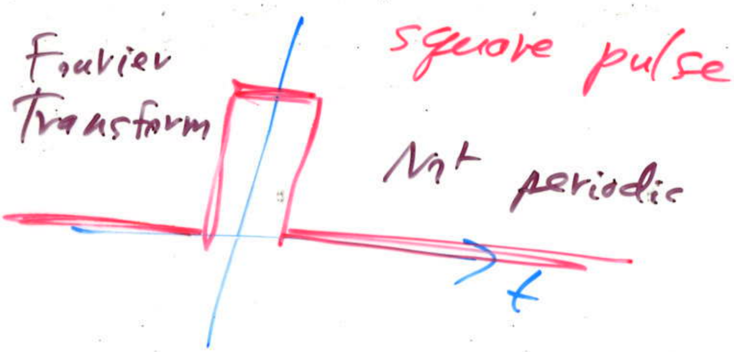
$\omega_0 = \frac{2\pi}{T}$

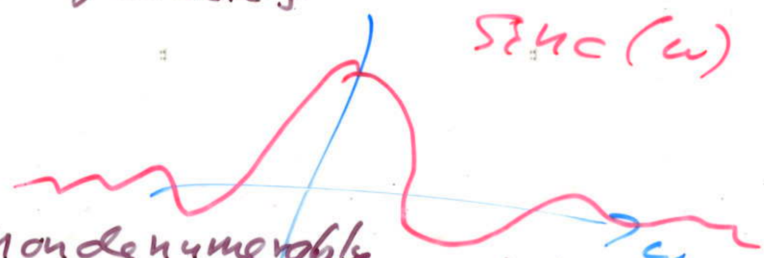


$\aleph_0$ – aleph null

denumerably infinitely many frequencies

Fourier Transform

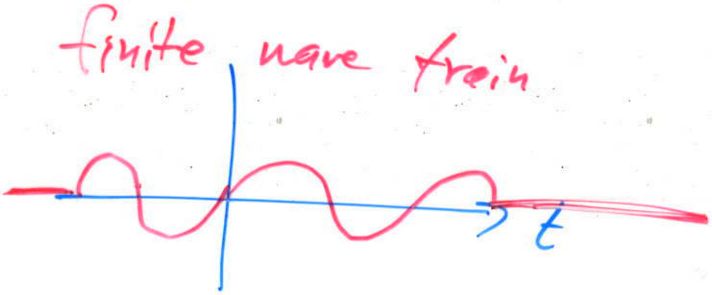


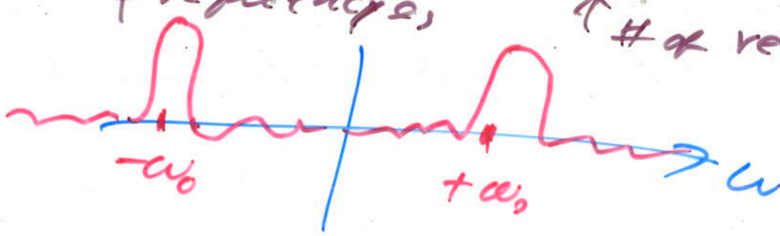


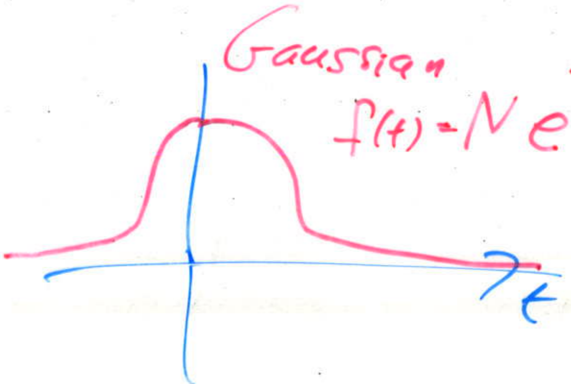
nondenumerably infinitely many frequencies

$\aleph_1 = 2^{\aleph_0}$

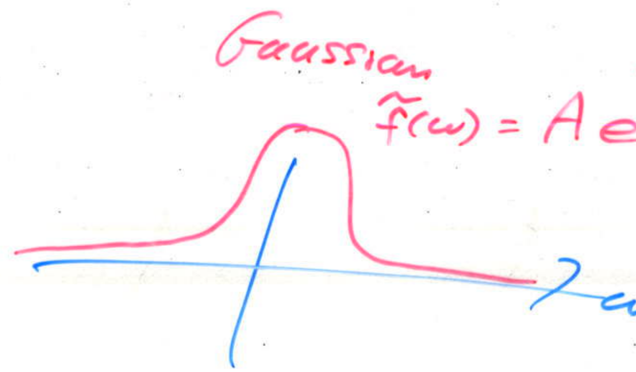
↑ # of reals







$f(t) = N e^{-\frac{t^2}{\tau^2}}$



$\tilde{f}(\omega) = A e^{-\frac{\omega^2}{\beta^2}}$