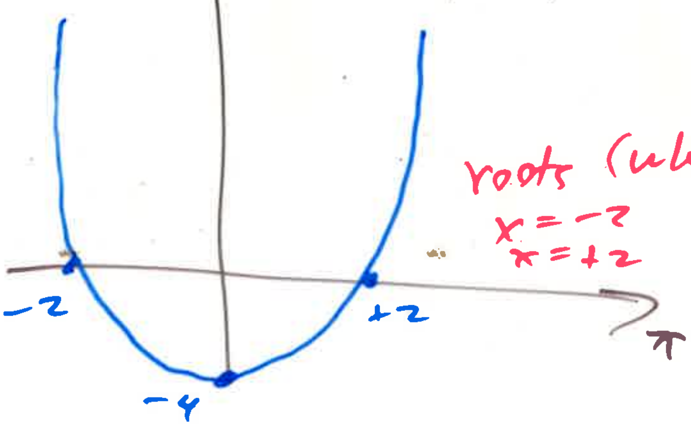
Complex numbers

Real number are not closed under algebraic operations.

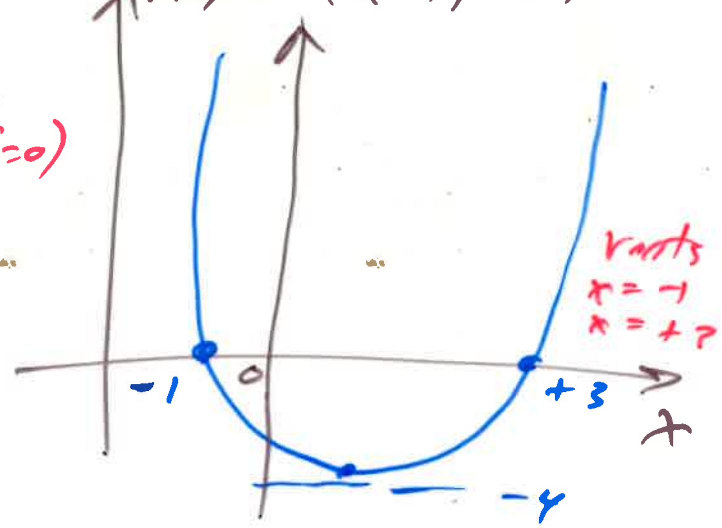
Quaternions : $\{\underline{\underline{I}}_{2\times 2}, \underline{\underline{\sigma}}_x, \underline{\underline{\sigma}}_y, \underline{\underline{\sigma}}_z\} = \{1, i, j, k\}$

$f(x) = x^2 - 4$



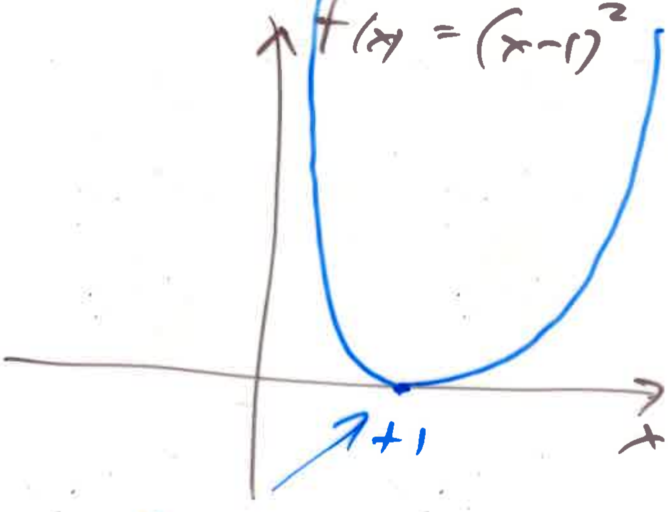
roots (where $f=0$)
$x = -2$
$x = +2$

$f(x) = (x-1)^2 - 4$



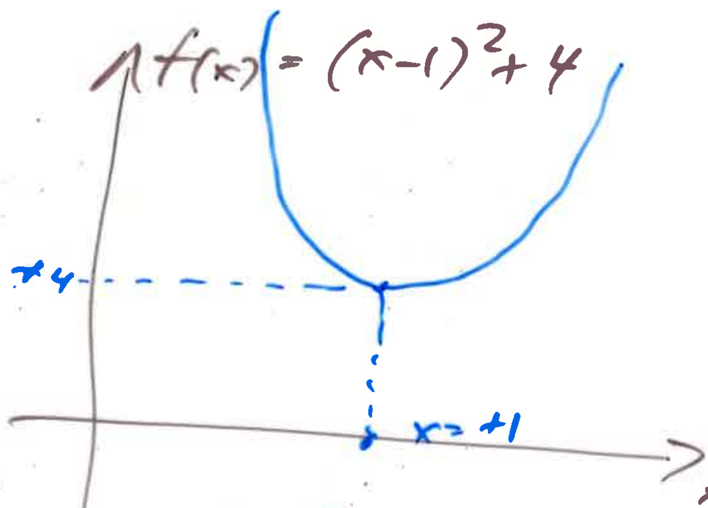
roots
$x = -1$
$x = +3$

$f(x) = (x-1)^2$



double root
roots
$x = 1$
$x = 1$

$f(x) = (x-1)^2 + 4$



roots
$x = 1 + 2i$
$x = 1 - 2i$

Quadratic Equation

$$f(x) = ax^2 + bx + c$$

roots: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Gerolamo Cardano, 1545 , Simon Stevin, 1594

$$z = \underbrace{\underset{\mathrm{Re}(z)}{x} + i\,\underset{\mathrm{Im}(z)}{y}}_{\text{Cartesian form}} = r e^{i\theta}$$

θ: $\arg(z) = \mathrm{ph}(z)$ — argument, phase

r: $\mathrm{mod}(z)$ — modulus (norm)

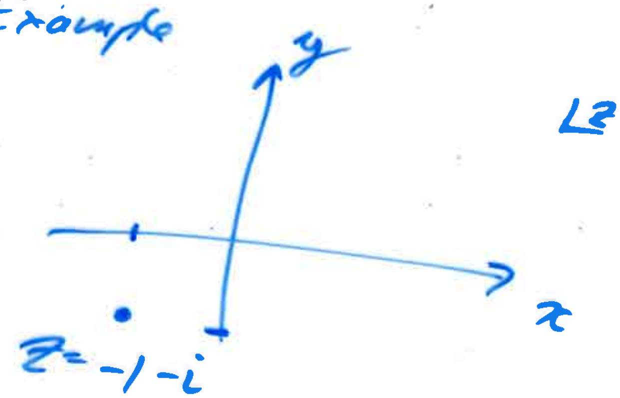
$$r = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right) \pm \pi$$ maybe

Example



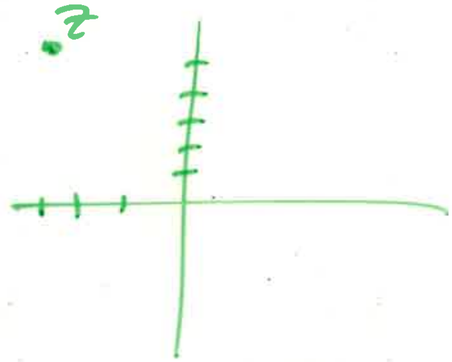
$x = -1$
$y = -1$
$r = \sqrt{2}$
$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

$$\sqrt[2]{-3+5i} = \sqrt[2]{z} = w = a+ib$$

$z = x+iy$; $x=-3$, $y=+5$

convert to polar form $r = \sqrt{x^2+y^2} = \sqrt{3^2+5^2}$

real number $= \sqrt{34}$



$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{5}{-3}\right)$$

$$= 2.111 \text{ rad} + 2\pi n$$

integer

$n = 0, \pm 1, \pm 2, \ldots$

$$z = r e^{i\theta}$$

$$w = \sqrt{z} = z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{\frac{i(\theta+2\pi n)}{2}}$$

$w_1 = r^{1/2} e^{\frac{i\theta}{2}}$ $(n=0)$ $a_1 = r^{1/2}\cos\left(\frac{\theta}{2}\right)$

$b_1 = r^{1/2}\sin\left(\frac{\theta}{2}\right)$

$w_2 = r^{1/2} e^{\frac{i}{2}(\theta+2\pi)}$ $(n=1)$ $a_2 = r^{1/2}\cos\left(\frac{\theta+2\pi}{2}\right)$

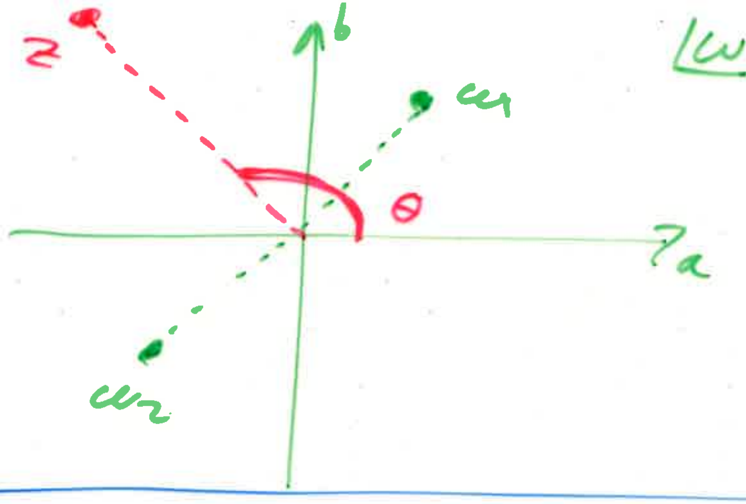
$b_2 = r^{1/2}\sin\left(\frac{\theta+2\pi}{2}\right)$

$$w_1 = a_1 + ib_1 = 2.415\cos\left(\frac{2.111}{2}\text{rad}\right) + i\,2.415\sin\left(\frac{2.111}{2}\text{rad}\right)$$

$$= 1.1901 + i\,2.104$$

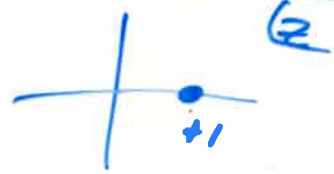
$$w_2 = a_2 + ib_2 = 2.415\cos\left(\frac{2.111+2\pi}{2}\right) + i\,2.415\sin\left(\frac{2.111+2\pi}{2}\right)$$

$$= -1.1901 - i\,2.104$$



$$\arg(w_1) = \tfrac{1}{2}\arg(z)$$
$$\arg(w_2) = \tfrac{1}{2}\arg(z) + \pi$$

Example : roots of unity



Find the 4 , 4^{th} roots of +1

$$Z = +1 = x+iy \Longrightarrow \begin{matrix} x=1 \\ y=0 \end{matrix} \;,\; r=1 \;;\; \theta = 0 + 2\pi n$$

$$w^4 = z \Rightarrow w = z^{1/4} = \left(r e^{i(\theta + 2\pi n)}\right)^{1/4}$$

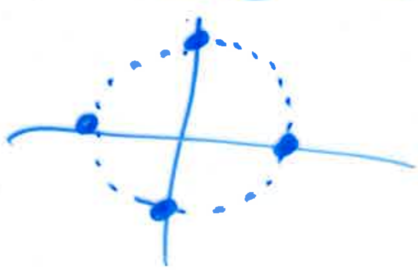
$$= \underbrace{r^{1/4}}_{1} e^{\frac{i(\theta+2\pi n)}{4}} = 1\, e^{i\frac{2\pi n}{4}}$$

$n=0$: $w_0 = 1$

$n=1$: $w_1 = 1\, e^{\frac{i\pi}{2}} = a = 1\cos(\frac{\pi}{2})^{0}$, $b = 1\sin(\frac{\pi}{2}) = 1 \,\big|\, w_1 = i$

$n=2$: $w_2 = 1\, e^{i\pi} = -1$ $(a=-1, b=0)$

$n=3$: $w_3 = 1\, e^{\frac{i3\pi}{2}} = -i$ $(a=0;\ b=-1)$



Matrices

Eigenvalues + eigenvectors

In general $\underline{\underline{M}}\,\vec{v} = \vec{w}$

linear transformation

clearest in a basis

$$\underset{(n \times n)}{\begin{pmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & m_{22} & \cdots \\ \vdots & & \vdots \\ m_{n1} & m_{n2} \cdots & m_{nn} \end{pmatrix}} \underset{(n \times 1)}{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_n \end{pmatrix}} = \underset{(n \times 1)}{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}}$$

first row: $m_{11} v_1 + m_{12} v_2 + m_{13} v_3 + \cdots + m_{1n} v_n = w_1$

There are special vectors $\vec{v}$ such that

$$\underline{\underline{M}}\,\vec{v}_i = \lambda_i \vec{v}_i$$

eigenvector

eigenvalue
number

Example

$$\underline{\underline{M}} = \begin{pmatrix} 6 & -4 \\ 5 & 2 \end{pmatrix}$$

$$\underline{\underline{M}}\,\vec{v} = \lambda \vec{v} = \lambda \underline{\underline{I}}_{2\times 2}\,\vec{v}$$

$$\underbrace{(\underline{\underline{M}} - \lambda \underline{\underline{I}})}_{\underline{\underline{P}}}\,\vec{v} = \vec{0} - \left.\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} n$$

$$\underline{\underline{P}}\,\vec{v} = \vec{0}$$

If $\underline{\underline{P}}$ has an inverse,

$$\underline{\underline{P}}^{-1} \cdot \underline{\underline{P}} = \underline{\underline{I}} = \underline{\underline{P}}\,\underline{\underline{P}}^{-1}$$

$$\underline{\underline{P}}^{-1}\underline{\underline{P}} \cdot \vec{v} = \underline{\underline{P}}^{-1} \cdot \vec{0}$$

$$\underline{\underline{I}}\,\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$$

Therefore $\nexists\, \underline{\underline{P}}^{-1}$, $\nexists\, (\underline{\underline{M}} - \lambda \underline{\underline{I}})$

Therefore $\det(\underline{\underline{M}} - \lambda \underline{\underline{I}}) = 0$

This will give eigenvalues λ

$$(\underline{\underline{M}} - \lambda \underline{\underline{I}}) = \begin{pmatrix} 6-\lambda & -4 \\ 5 & 2-\lambda \end{pmatrix}$$

$$\det(\underline{\underline{M}} - \lambda \underline{\underline{I}}) = |\underline{\underline{M}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} 6-\lambda & -4 \\ 5 & 2-\lambda \end{vmatrix}$$

$$(6-\lambda)(2-\lambda) - (5)(-4) = 0$$

$$12 - 8\lambda + \lambda^2 + 20 = 0$$

$\lambda^2 - 8\lambda + 32 = 0$

$a = 1$
$b = -8$
$c = 32$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{-64}}{2}$$

$\lambda = 4 \pm 4i$ $\quad \lambda_1 = 4+4i \quad , \lambda_2 = 4-4i$

Now find the eigenvectors $\vec{v}_1, \vec{v}_2$:

$$(\underline{\underline{M}} - \lambda \underline{\underline{I}})\vec{v} = \vec{0}$$

① $$(\underline{\underline{M}} - \lambda_1 \underline{\underline{I}})\vec{v}_1 = \begin{pmatrix} +2-4i & -4 \\ 5 & -2-4i \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

top row: $(2-4i)v_a - 4v_b = 0$

say $v_b = 1 \Rightarrow (2-4i)v_a = 4$

$v_a = \frac{4}{2-4i}$

$$\vec{v}_1 = \begin{pmatrix} \frac{4}{2-4i} \\ 1 \end{pmatrix}$$

first eigenvector with eigenvalue $\lambda_1 = 4+4i$

With Mathematica

```
In[1]:= m = {{6, -4}, {5, 2}};
        m // MatrixForm
```

Out[2]//MatrixForm=
$$\begin{pmatrix} 6 & -4 \\ 5 & 2 \end{pmatrix}$$

```
In[3]:= Eigensystem[m]
```

Out[3]=
$$\Big\{\{\underbrace{4+4\,i}_{\lambda_1},\ \underbrace{4-4\,i}_{\lambda_2}\},\ \Big\{\underbrace{\Big\{\frac{2}{5}+\frac{4\,i}{5},\ 1\Big\}}_{\vec{v}_1},\ \underbrace{\Big\{\frac{2}{5}-\frac{4\,i}{5},\ 1\Big\}}_{\vec{v}_2}\Big\}\Big\}$$

Our $\vec{v}_1 = \begin{pmatrix} \frac{4}{2-4i} \\ 1 \end{pmatrix}$ looks different from the $\vec{v}_1$ above, but they are the same.

(Multiply $\frac{4}{2-4i}$ by $\frac{2+4i}{2+4i}$).