$$f(x) = \psi(x) = A_0 \sin(kx) + B_0 \cos(kx)$$

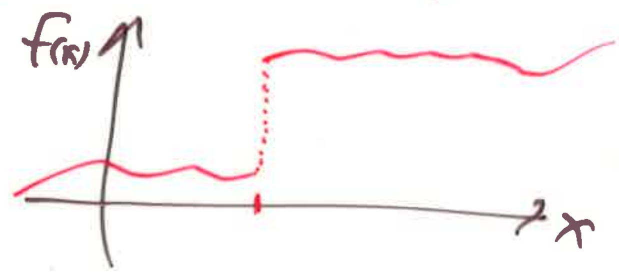
$\uparrow$ space part of $\Psi(x,t) = f(x)\, g(t)$ $\qquad k^2 = \frac{2mE}{\hbar^2}$

$$g(t) = g_0\, e^{-\frac{iEt}{\hbar}}$$

Fix A_0 and B_0 with boundary conditions

$f(x)$ must be continuous (draw $f(x)$ without removing pen from paper

Not



no vertical lines because then $f(x)$ is not a function.

Show this by integrating the S.E.

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} [\,\ldots\, SE\,]\, dx \longrightarrow \Delta\psi = \psi(+\epsilon) - \psi(-\epsilon) = 0$$

B.C. $f(0) = 0$

$f(a) = 0$

If $f(x) = 0$ everywhere, then it can't be normalized

$$\int_{-\infty}^{\infty} f^*(x)\, f(x)\, dx = 1 \implies 0 = 1$$

$$f(x) = A_0 \sin(kx) + B_0 \cos(kx)$$

$$f(0) = A_0 \underbrace{\cancel{\sin(k0)}}_{\to 0} + B_0 \underbrace{\cancel{\cos(k0)}}_{\to 1} = \boxed{B_0 = 0}$$

$$f(x) = A_0 \sin(kx)$$

$$f(a) = A_0 \sin(ka) = 0 \qquad k_n = \frac{n\pi}{a}$$

$$\Rightarrow \sin(ka) = 0 \rightarrow ka = n\pi$$

for integer $\underline{n}$

$n = 0$ not allowed, because then $f(x) = 0 \ \forall x$

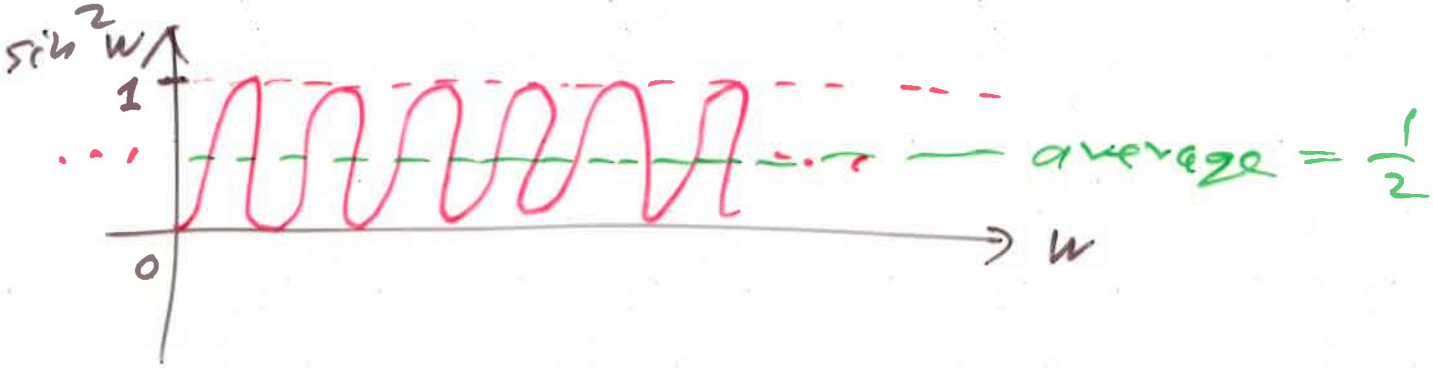
$n < 0$ duplicates $n > 0$. $\qquad n = 1, 2, 3, \ldots$

$$f_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, 3 \ldots$$

Determine A_n by normalizing $\Psi(x,t)$

$$1 = \int_{x=-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx = \int_{x=0}^{a} f_n^*(x)\, f_n(x)\, dx$$

$$\int_{x=0}^{a} A_n^* A_n \sin^2\left(\frac{n\pi x}{a}\right) dx = |A_n|^2 \underbrace{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx}_{\frac{a}{2}}$$



$$1 = |A_n|^2 \frac{a}{2} \implies |A_n|^2 = \frac{2}{a}$$

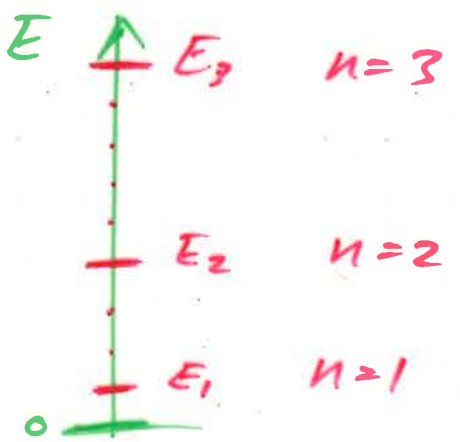
Can choose A_n real $\implies A_n = \sqrt{\frac{2}{a}}$.

Could also have chosen $A_n = \sqrt{\frac{2}{a}}\, e^{i\theta}$

Right now we have enough information to determine the quantized energy levels of the particle in a 1-dim box.

$$k^2 = \frac{2mE}{\hbar^2} \implies E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

$n = 1, 2, 3, \ldots$ (not 0)

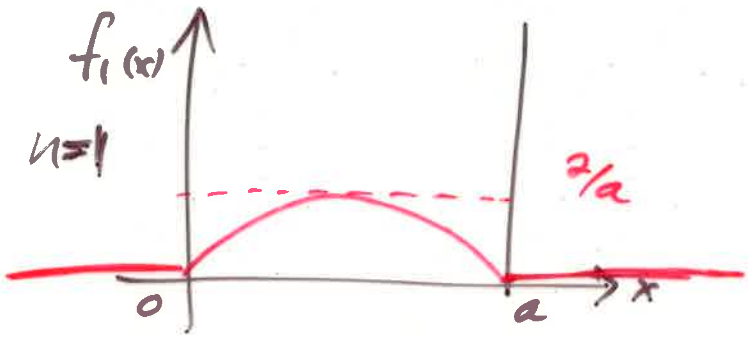


$\boxed{\text{Box} \quad E_n \propto n^2}$

SHO $\quad E_n \propto n$

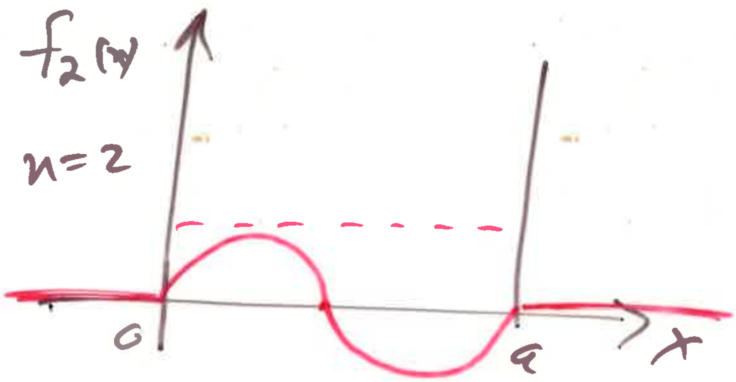
H-atom $\quad E_n \propto \frac{1}{n^2}$

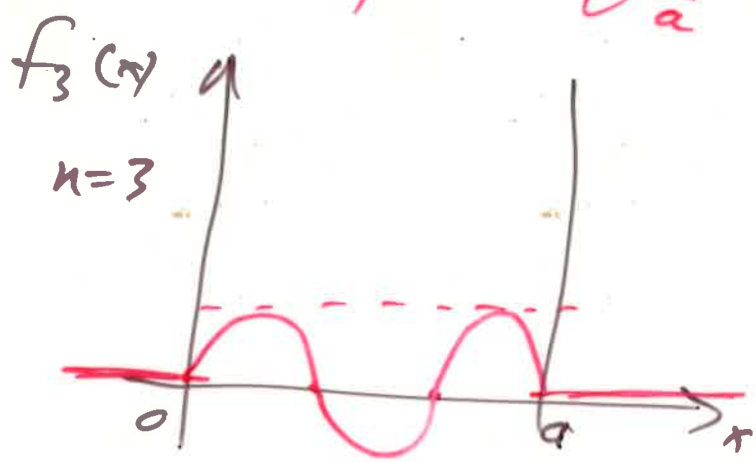
$$f_n(x) = A_0 \sin(k_n x) = \underline{\sqrt{\frac{2}{a}}} \sin\left(\frac{n\pi x}{a}\right)$$



$$g(t) = 1\, e^{-\frac{iEt}{\hbar}}$$

f_n all have same amplitude $\sqrt{\frac{2}{a}}$





The wavefunctions $f_n(x)$ are orthonormal.

$$|f_n\rangle = f_n(x)$$

like $\hat{u}_1 \cdot \hat{u}_1$

$$\langle f_n | f_n \rangle = \int_{\cancel{-\infty}\;0}^{\cancel{\infty}\;a} f_n^*(x) f(x)\, dx = 1$$

$$\langle f_n | f_p \rangle = \int_{-\infty}^{\infty} f_n^*(x)\, f_p(x)\, dx$$

$$= \frac{2}{a} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{p\pi x}{a}\right) dx$$

$$\sin\alpha \sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

$$\langle f_n | f_p \rangle = \begin{cases} 1, & n=p \\ 0, & n \neq p \end{cases} = \delta_{np}$$

Any function $R(x)$ that vanishes at $x=0$ and $x=a$ can be expanded in $f_n(x)$.

$$R(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

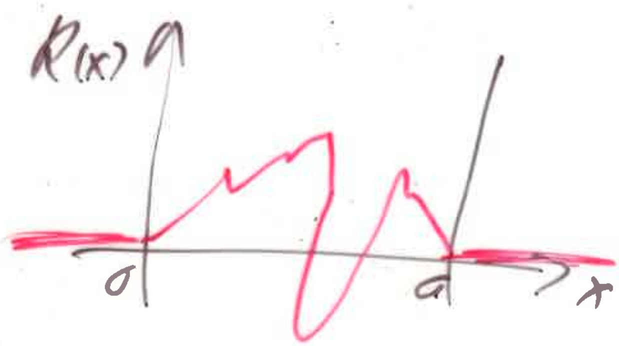
closure = completeness

orthonormal basis "vectors"

complex coefficients

$$R(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

Fourier sine series



$$c_n = \langle f_n | R(x) \rangle$$

Looks like $\vec{F} = \sum_{n=1}^{3} b_n \hat{u}_n \Rightarrow b_n = \hat{u}_n \cdot \vec{F}$

$$c_n = \int_{x=\cancel{-\infty}\ 0}^{\cancel{\infty}\ a} f_n^*(x) R(x)\, dx$$

$$c_n = \int_{x=0}^{a} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) R_{(x)}\, dx$$

Most general wavefunction $\Psi_{(x,t)}$ solution to the infinite square well.

$$\Psi_{(x,t)} = \sum_{n=1}^{\infty} \underset{\text{complex numbers}}{c_n} \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)}_{f_n(x)} \underbrace{\exp\left[\frac{-it}{\hbar}\overbrace{\left(\frac{n^2\pi^2\hbar^2}{2ma^2}\right)}^{E_n}\right]}_{g_n(t)}$$

$\Psi_{(x,t)}$ must also be normalized

$$\int_{x=-\infty}^{+\infty} \Psi^*_{(x,t)} \Psi_{(x,t)}\, dx = 1 \Longrightarrow \sum_{n=1}^{\infty} |c_n|^2 = 1$$

A particle of mass m in box with wavefunction $\Psi_{(x,t)}$ as above does not have <u>one</u> energy, but you can find the expectation value of the energy $\langle H \rangle = \int_{-\infty}^{\infty} \Psi^*_{(x,t)}\, i\hbar\frac{\partial}{\partial t} \Psi_{(x,t)}\, dx$