$$\Psi_n(x,t) = f_n(x)\, g_n(t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \exp\left[\frac{-it}{\hbar} E_n\right]$$

does have a unique energy $= E_n$

examples $c_1 = \sqrt{\frac{1}{2}} = c_2$, all others $c_n = 0$

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electron confined to $1 \mathring{A} = 10^{-10}$ m

$$E_1 = 6.0 \times 10^{-18} \text{ J}$$

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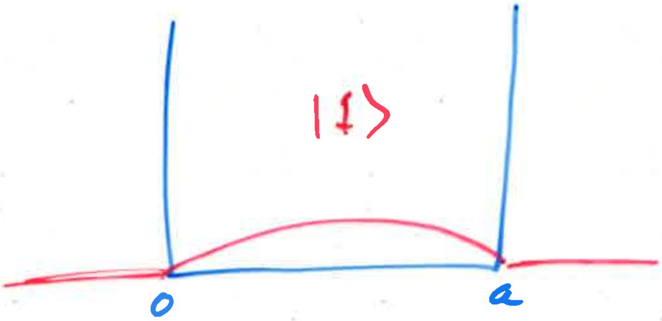
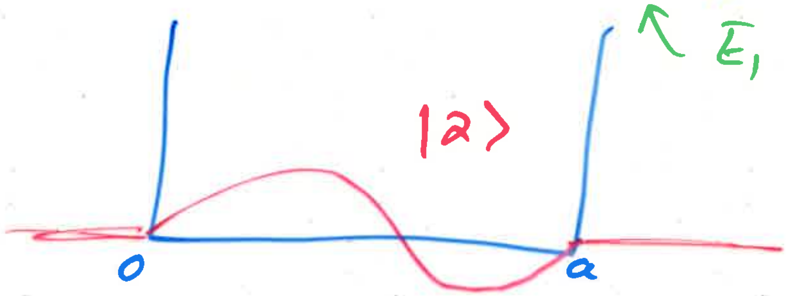
1 kg confined to 7 m

$$E_1 = 1.1 \times 10^{-69} \text{ J}$$

$$|1\rangle \equiv \sqrt{\frac{2}{a}}\,\sin\left(\frac{\pi x}{a}\right)\exp\left[\frac{-it}{\hbar}\underbrace{\left(\frac{\pi^2\hbar^2}{2ma^2}\right)}_{E_1}\right]$$





$$|2\rangle \equiv \sqrt{\frac{2}{a}}\,\sin\left(\frac{2\pi x}{a}\right)\exp\left[\frac{-it}{\hbar}\underbrace{\left(\frac{4\pi^2\hbar^2}{2ma^2}\right)}_{E_2}\right]$$

$$E_1 = \hbar\omega_1 = h\nu_1 = hf_1 \qquad \hbar \equiv \frac{h}{2\pi}$$

angular frequency — MKS → rad/sec

linear frequency — $Hz = \frac{\text{cycles}}{\text{sec}}$

$\omega = 2\pi f$
$= 2\pi\nu$

Like guitar string $\lambda_2 = \frac{1}{2}\lambda_1$ λ = wavelength

Unlike a guitar string: $\nu_2 = 4\nu_1$
$\nu_3 = 9\nu_1$

Really unlike a guitar string $E_2 = 4E_1$
$E_3 = 9E_1$

guitar string has *any* energy you like.

$$\langle 1|1\rangle = 1 = \int_{x=\cancel{-\infty}\,0}^{\cancel{\infty}\,a} \sqrt{\tfrac{2}{a}}^{\,*}\sin\left(\tfrac{\pi x}{a}\right)^{*} e^{\frac{+itE_1}{\hbar}} \sqrt{\tfrac{2}{a}}\sin\left(\tfrac{\pi x}{a}\right) e^{\frac{-itE_1}{\hbar}}\,dx$$

$$= \frac{2}{a}\int_{x=0}^{a} \sin^2\left(\frac{\pi x}{a}\right)dx = 1 \quad \checkmark$$

$$\langle 2|1\rangle = 0 \qquad\qquad \langle 2|2\rangle = 1$$

$$\langle n|p\rangle = \delta_{np}$$

$|1\rangle, |2\rangle, |3\rangle \ldots |n\rangle \ldots$ basis vectors for Hilbert space

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Linear Superposition

$$|\Psi\rangle \equiv \Psi_{(x,t)} = \sum_{n=1}^{\infty} c_n |n\rangle \quad \text{in general}$$

special combination

$$|\Psi\rangle = \underbrace{\frac{1}{\sqrt{2}}}_{c_1}|1\rangle + \underbrace{\frac{1}{\sqrt{2}}}_{c_2}|2\rangle$$

$c_1 = \frac{1}{\sqrt{2}} = c_2$

all other $c_n = 0$

restriction on coefficients: $\sum_{n=1}^{\infty} |c_n|^2 = 1$

$$|c_1|^2 + |c_2|^2 = \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

$|\Psi\rangle$ is normalized

$$\langle\Psi|\Psi\rangle = \left(\frac{1}{\sqrt{2}}^* \langle 1| + \frac{1}{\sqrt{2}}^* \langle 2|\right)\left(\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle\right)$$

$$= \frac{1}{2}\underbrace{\langle 1|1\rangle}_{1} + \frac{1}{2}\underbrace{\langle 2|2\rangle}_{1} + \frac{1}{2}\underbrace{\langle 1|2\rangle}_{0} + \frac{1}{2}\underbrace{\langle 2|1\rangle}_{0} = 1$$

Expectation Value of the energy → Hamiltonian operator

$$\langle\hat{H}\rangle = \langle\Psi|\hat{H}|\Psi\rangle = \langle\Psi| i\hbar\frac{\partial}{\partial t}|\Psi\rangle$$

$$\underset{\text{matrix}}{\hat{H}}\ \underset{\text{eigen vector}}{|1\rangle} = i\hbar\frac{\partial}{\partial t}|1\rangle = \underset{\text{eigen value}}{E_1}\ \underset{\text{eigen vector}}{|1\rangle}$$

In this basis $|1\rangle, |2\rangle, |3\rangle, \ldots |n\rangle$

$$\hat{H} = \begin{pmatrix} E_1 & 0 & 0 & & \\ 0 & E_2 & 0 & & \\ 0 & 0 & E_3 & & \\ & & & \ddots & \\ & & & & E_n \\ & & & & & \ddots \end{pmatrix} \quad \text{diagonal matrix}$$

$$H_{np} = \langle n|\hat{H}|p\rangle = E_n\delta_{np}$$

$$\hat{H}|2\rangle = E_2|2\rangle \qquad \hat{H}|n\rangle = E_n|n\rangle$$

The states $|1\rangle, |2\rangle \ldots |n\rangle \ldots$

are eigenstates (eigenkets) of the Hamiltonian operator. They are also called stationary states. They have definite energy E_n.

$$\langle \Psi | \hat{A} | \Psi \rangle = \left(\tfrac{1}{\sqrt{2}}^* \langle 1| + \tfrac{1}{\sqrt{2}}^* \langle 2|\right) \hat{A} \left(\tfrac{1}{\sqrt{2}}|1\rangle + \tfrac{1}{\sqrt{2}}|2\rangle\right)$$

$$= \frac{1}{2} E_1 \langle 1|1\rangle + \frac{1}{2} E_2 \langle 2|2\rangle + \frac{1}{2} E_1 \cancel{\langle 2|1\rangle}^{0} + \frac{1}{2} E_2 \cancel{\langle 1|2\rangle}^{0}$$

$$= \underline{\frac{1}{2}(E_1 + E_2)} = \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right)(1+4) = \frac{5}{2}\frac{\pi^2\hbar^2}{2ma^2}$$

Meaning of expectation value

Prepare a large number ~1 billion identical systems, measure the energy.

~500 million you will get E_1

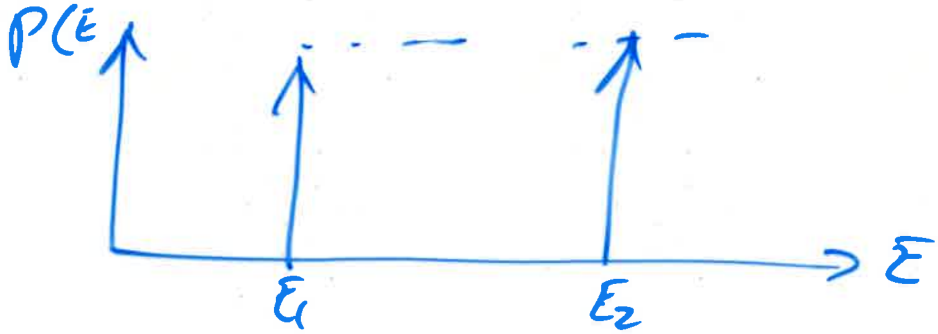
~500 million you will get E_2

Average is $\dfrac{500 \text{ million } E_1 + 500 \text{ million } E_2}{1 \text{ billion}} = \dfrac{E_1 + E_2}{2}$

but **NONE** of the measurements will ever give $\dfrac{5}{2}\dfrac{\pi^2\hbar^2}{2ma^2}$

Most likely energy measurement?

E_1 and E_2 are both equally likely



After a measurement in which E_1 is obtained, the wavefunction collapses to $|1\rangle$ only. Stays $|1\rangle$ forever.

Measure energy and get E_2, the wavefunction is pure $|2\rangle$. Stays $|2\rangle$ forever.

Measure x

$x|1\rangle$

Expectation value

$$\langle 1|x|1\rangle = \frac{a}{2}$$
$$\langle 2|x|2\rangle = \frac{a}{2}$$
$$\langle 1|x|2\rangle = ?$$
$$\langle \Psi|x|\Psi\rangle = ?$$

expect "beats"

sinusoidal time dependence