$$\text{S.E.}\quad i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + \underbrace{V(x)\,\Psi(x,t)}_{\frac{1}{2}m\omega^2x^2\Psi(x,t)}$$

2nd-order, linear in Ψ, homogeneous PDE.

Ansatz: $\Psi(x,t) = f(x)\cdot g(t)$ separation of variables

$$g(t) = \underbrace{g_0 \exp\left[\frac{-it}{\hbar}E\right]}_{\text{same as before}}$$

E is different.

ODE for x:

$$\frac{-\hbar^2}{2m}\frac{d^2f(x)}{dx^2} + \frac{1}{2}m\omega^2x^2f(x) = Ef(x)$$

$$f''(x) - \left(\frac{m^2\omega^2x^2}{\hbar^2} - \frac{2mE}{\hbar^2}\right)f(x) = 0$$

2nd-order, linear (in f), homogeneous ODE.
with non-constant coefficients.

Now what? A) Look it up in a book

B) Symbolic Manipulator

~~C) Solve it yourself.~~

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - + \cdots$$

$\uparrow$ dimensionless

The answer

$$|n\rangle = f(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\Big(\underbrace{\sqrt{\tfrac{m\omega}{\hbar}}\,x}_{z}\Big)\, e^{-\frac{1}{2}\overbrace{\frac{m\omega}{\hbar}x^2}^{z^2}}$$

H_n are Hermite polynomials

$H_0(z) = 1$

$H_1(z) = 2z$

$H_2(z) = 4z^2 - 2$

2^n power n

$H_3(z) = 8z^3 - 12z$

$H_4(z) = 16z^4 - 48z^2 + 12$

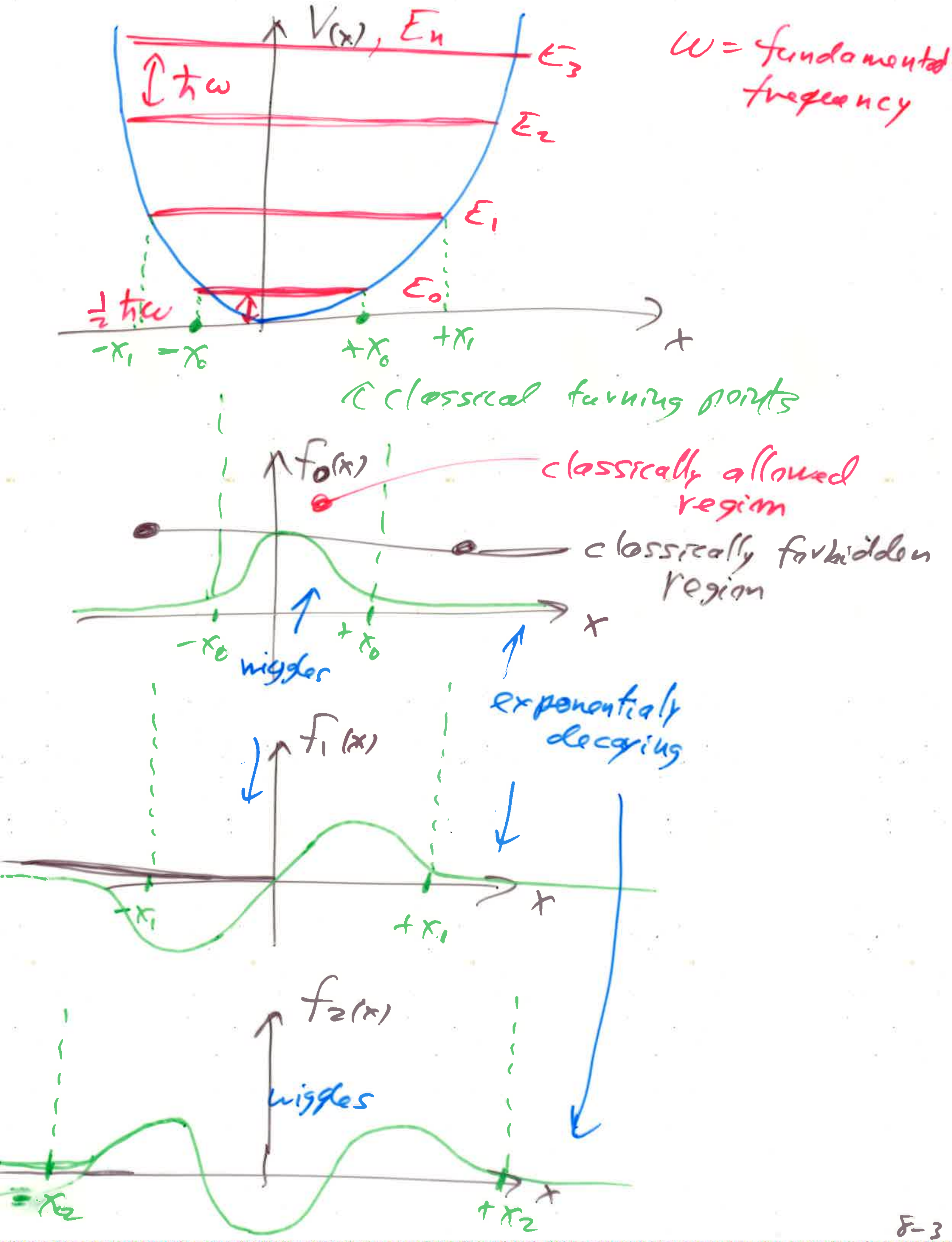
$H_\vdots$

orthogonality

$$\int_{-\infty}^{\infty} e^{-z^2} H_n(z) H_p(z)\, dz = \delta_{np} \sqrt{\pi}\, n!\, 2^n$$

$$g(t) = \cancel{g_0}^{\,1} \exp\left(\frac{-it}{\hbar} E_n\right) = \exp\left[\frac{-it}{\hbar}\left(n+\tfrac{1}{2}\right)\hbar\omega\right]$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, 3 \ldots$$



V(x), E_n
ħω
E_3
E_2
E_1
E_0
½ ħω
−x_1
−x_0
+x_0
+x_1
x
ω = fundamental frequency
← classical turning points
f_0(x)
classically allowed region
classically forbidden region
wiggles
exponentially decaying
f_1(x)
f_2(x)
−x_2
+x_2

1-dimension time-independent SE

$$\frac{d^2 f(x)}{dx^2} = \frac{2m}{\hbar^2}\left[V(x) - E\right] f(x)$$

concavity > 0 < 0

if > 0 < 0

$\Rightarrow$ In the classically forbidden region $V(x) > E$ then no wiggles

See Griffiths Problem 2.2

Commutator of 2 quantum operators (not states)

e.g. $\hat{x}$, $\hat{p}$, $\hat{H}$, $\hat{x}^2$, $\hat{p}^2$ --- $\uparrow \psi$

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

operator

$$[\hat{A}, \hat{B}]\psi = \hat{A}(\hat{B}\psi) - \hat{B}(\hat{A}\psi)$$

Look at the most famous commutation relation

$$[\hat{x}, \hat{p}_x] = ?$$

Essence of Quantum Mechanics

$$[\hat{x}, \hat{p}_x]\psi(x)$$

$$\left.\begin{aligned} \hat{x} &= x \\ \hat{p}_x &= \frac{\hbar}{i}\frac{\partial}{\partial x} \end{aligned}\right\}$$ in coordinate basis.

$$= x\left(\frac{\hbar}{i}\frac{\partial \psi(x)}{\partial x}\right) - \frac{\hbar}{i}\frac{\partial}{\partial x}\left[x\,\psi(x)\right]$$

$$= \frac{\hbar}{i}\left[x\frac{d\psi(x)}{dx} - 1\,\psi(x) - x\frac{d\psi}{dx}\right] = -\frac{\hbar}{i}1\,\psi(x)$$

$$\Longrightarrow [\hat{x}, \hat{p}_x]\psi(x) = i\hbar\,\psi(x)$$

$$\boxed{[\hat{x}, \hat{p}_x] = i\hbar} \Longrightarrow \Delta x \Delta p_x \geq \frac{\hbar}{2}$$

Heisenberg Uncertainty

$$[\hat{y}, \hat{p}_y] = i\hbar \qquad [\hat{z}, \hat{p}_z] = i\hbar$$

$$[\hat{x}, \hat{y}] = 0 = [\hat{y}, \hat{z}] = [\hat{z}, \hat{x}]$$

$$[\hat{p}_x, \hat{p}_y] = 0 = \quad \text{— — —}$$

$$[\hat{x}, \hat{p}_y] = 0 \qquad [\hat{y}, \hat{p}_x] = 0 \qquad [\hat{y}, \hat{p}_z]$$

$$[\hat{H}, ?] \longrightarrow \frac{d}{dt} ?$$