Ladder Operators

a.k.a. Raising + Lower Ops

Creation + Annihilation ops

Harmonic Oscillator

S.E. $i\hbar \frac{\partial}{\partial t}\Psi_{(x,t)} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_{(x,t)} + V_{(x)}\Psi_{(x,t)}$

$$\hat{H}\Psi_{(x,t)} = \frac{\hat{P}_x^{\,2}}{2m}\Psi_{(x,t)} + \frac{1}{2}\underbrace{m\omega^2}_{k}\hat{x}^2\Psi_{(x,t)}$$

↑ Hamiltonian

$$\hat{H} = \frac{1}{2m}\left[\hat{P}_x^{\,2} + (m\omega\hat{x})^2\right]$$

$\hat{x} = x$

$\hat{P}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Define

Raising Op.

$$\hat{a}_+ \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{P}_x + m\omega\hat{x}\right) \equiv \hat{a}^\dagger$$

creation op

need to act on a state.

Lowering Op.

$$\hat{a}_- \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{P}_x + m\omega\hat{x}\right) \equiv \hat{a}$$

annihilation op.

$(\hat{a}_-)^\dagger = \hat{a}_+ \quad , \quad (\hat{a}_+)^\dagger = \hat{a}_-$

$\dagger$ = dagger = hermitian conjugate

= transpose, complex conjugate

$\dagger = T^*$

$$\begin{pmatrix} 1 & 2 \\ i & 3i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & -i \\ 2 & -3i \end{pmatrix}$$

Consider $\hat{a}_- \hat{a}_+ \Psi \equiv \hat{a}_- (\hat{a}_+ \Psi)$ — order matters

$$\hat{a}_- \hat{a}_+ = \frac{1}{2\hbar m\omega}\left(i\hat{p}_x + m\omega\hat{x}\right)\left(-i\hat{p}_x + m\omega\hat{x}\right)$$

$$= \frac{1}{2\hbar m\omega}\left(\hat{p}_x^2 + m^2\omega^2\hat{x}^2 - im\omega\underbrace{[\hat{x}\hat{p}_x - \hat{p}_x\hat{x}]}_{[\hat{x},\hat{p}_x] = i\hbar}\right)$$

$$= \frac{1}{\hbar\omega}\left\{\underbrace{\frac{1}{2m}\left(\hat{p}_x^2 + m^2\omega^2\hat{x}^2\right)}_{\hat{H}} + \frac{1}{2}\hbar\omega\right\}$$

$$= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}$$

$$\hat{a}_+ \hat{a}_- = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}$$ check.

$$[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2} - \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right) = 1$$

$$[\hat{a}_+, \hat{a}_-] = -1$$ commutator is antisymmetric

Solve for Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

$$\hat{H}\Psi_n = E_n\Psi_n$$

- $\hat{H}$: matrix op
- E_n: number, eigenvalue
- Ψ_n: eigenvector; wave function, state, ket, vector

$$\underline{\underline{M}}\,\vec{v} = \lambda\vec{v}$$

$$\underline{\underline{M}}\,(2\vec{v}) = \lambda(2\vec{v})$$

Look at the new state $\hat{a}_+\psi_n$. What is energy?

$$\hat{H}(\hat{a}_+\psi_n) = \left[\hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})\right](\hat{a}_+\psi_n)$$

expect (Energy) $(\hat{a}_+\psi_n)$

$$= \hbar\omega(\hat{a}_+\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\hat{a}_+)\psi_n$$

Use: $\hat{a}_-\hat{a}_+ = 1 + \hat{a}_+\hat{a}_-$

$$= \hbar\omega\hat{a}_+(\hat{a}_-\hat{a}_+ + \tfrac{1}{2})\psi_n$$

$$= \hbar\omega\hat{a}_+\left[(\hat{a}_+\hat{a}_- + 1) + \tfrac{1}{2}\right]\psi_n$$

$$= \hat{a}_+(\hat{H} + \hbar\omega)\psi_n = \hat{a}_+(E_n + \hbar\omega)\psi_n$$

$$= (E_n + \hbar\omega)(\hat{a}_+\psi_n)$$

If ψ_n has energy E_n, then $(\hat{a}_+\psi_n)$ has energy $(E_n + \hbar\omega)$ – one rung higher.

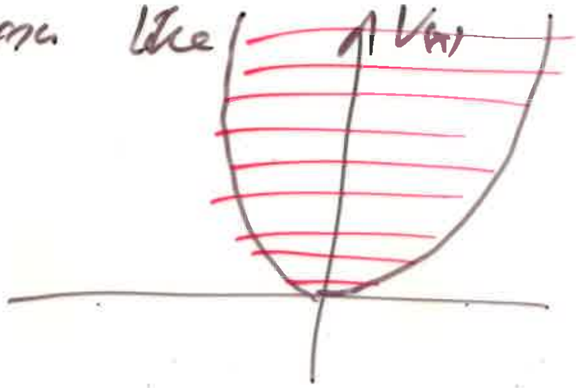
Warning: $(\hat{a}_+\psi_n)$ is not normalized!

At the same, check

$$\hat{H}(\hat{a}_-\psi_n) = \cdots = (E_n - \hbar\omega)(\hat{a}_-\psi_n)$$

$(\hat{a}_-\psi_n)$ also not normalized!

There is no upper limit to this procedure. Start with ψ_n that has energy E_n and apply raising operator $\hat{a}_+$ as many times as you like



not normalized!

states: ψ_n, $\hat{a}_+\psi$, $\hat{a}_+^2\psi$ ...

energy: E_n, $E_n+\hbar\omega$, $E_n+2\hbar\omega$ + ...

There is, however, a lower limit. There is a ground state (state of lowest energy) ψ_0.

ψ_0 ... $\hat{a}_-^2\psi$, $\hat{a}_-\psi_n$, ψ_n ⟶

E_0 ..., $E_n-2\hbar\omega$, $E_n-\hbar\omega$, E_n ⟶

Then what? $\hat{a}_-\psi_0 = \emptyset$

$$\hat{a}_-\psi_0(x) = \frac{1}{\sqrt{2m\hbar\omega}}\left(i\hat{p}_x + m\omega\hat{x}\right)\psi_0(x) = 0$$

$$\Longrightarrow \left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0(x) = 0$$

$$\frac{d\psi_0(x)}{dx} = -\frac{m\omega}{\hbar}x\,\psi_0(x)$$

first-order, linear homogeneous O.D.E.

separable.

$$\frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar} x \, dx$$

integrate both sides

$$\int_{\psi_0(0)}^{\psi_0(x)} \frac{d\psi_0'}{\psi_0'} = -\frac{m\omega}{\hbar} \int_0^x x' \, dx'$$

upper & lower limit must correspond

↑ x' is a dummy variable

$$\int_0^x y \, dy$$

$$\ln(\psi_0')\Big|_{\psi_0(0)}^{\psi_0(x)} = -\frac{m\omega}{\hbar} \frac{1}{2}(x')^2 \Big|_0^x = -\frac{m\omega}{2\hbar} x^2$$

$$\|$$

$$\ln[\psi_0(x)] - \underbrace{\ln[\psi_0(0)]}_{B} = \frac{-m\omega x^2}{2\hbar}$$

$$\ln[\psi_0(x)] = B - \frac{m\omega x^2}{2\hbar}$$

exponentiate both sides

$$\psi_0(x) = e^{B - \frac{m\omega x^2}{2\hbar}} = \underbrace{e^{B}}_{A} \cdot e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\psi_0(x) = A e^{-\frac{m\omega x^2}{2\hbar}}$$

choose A so that $\psi_0(x)$ is normalized

$$1 = \int_{x=-\infty}^{+\infty} |\psi_0|^2 dx = \int_{x=-\infty}^{+\infty} |A|^2 e^{\frac{-m\omega x^2}{\hbar}} dx$$ Gaussian integral

$$1 = |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$$ choose A to be real

$$\Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$

What is the energy?

Hard way: S.E. $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_0(x) + \underbrace{\frac{1}{2}m\omega^2 x^2}_{V(x)}\psi_0(x) = E_0\psi_0(x)$

take 2 derivatives... algebra ---

Easy way: $\hat{H}\psi_0(x) = E\psi_0(x)$

$$\downarrow$$

$$\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi_0(x) = E\psi_0(x)$$

$$= \hbar\omega\,\hat{a}_+\cancelto{0}{(\hat{a}_-\psi_0(x))} + \frac{1}{2}\hbar\omega\,\psi_0(x) = E_0\,\psi_0(x)$$

$$E_0 = \frac{1}{2}\hbar\omega\ !$$