Georg Cantor's Diagonalization Proof

that there are more reals than integers
$0 \le x \le 1$

Label

0) 0.7921459275...

1) 0.3233 3333...

2) 0.2323231111100...

3) 0.577952.....

4)

5)

$\aleph_0 = \infty$)

Missed: 0.6245... (For example)

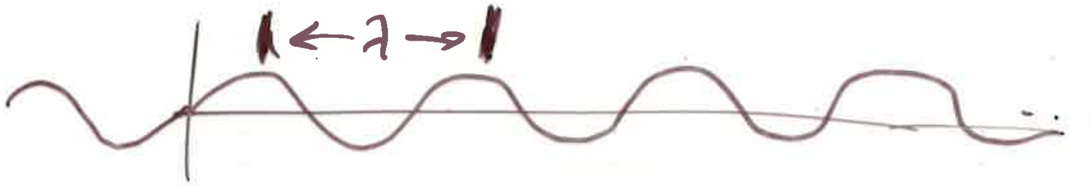
$\Longrightarrow$ More Real numbers than integers

c — for continuum $\qquad \aleph_0$

$$\text{c} \equiv \aleph_1 = 2^{\aleph_0}$$

k is the wavenumber $= |\vec{k}|$
wave vector

$k = \frac{2\pi}{\lambda}$



Linear Momentum $p = \hbar k = \frac{h}{\lambda}$ (de Broglie)

Two speeds

$\frac{\omega}{k}$ = phase velocity v_φ, $\frac{d\omega}{dk}$ = group velocity v_g

Angular frequency $\omega = \frac{E}{\hbar} \Rightarrow \omega = 2\pi f = 2\pi \nu$

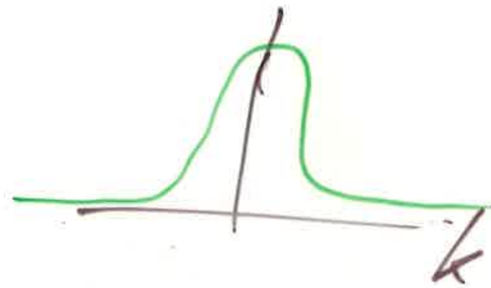
$\omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar} = \frac{\hbar k^2}{2m} = \omega(k)$ dispersion relation

$v_\varphi = \frac{\omega}{k} = \frac{\hbar k}{2m}$, $v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = 2 v_\varphi$

Fourier Transform introduce to QM!

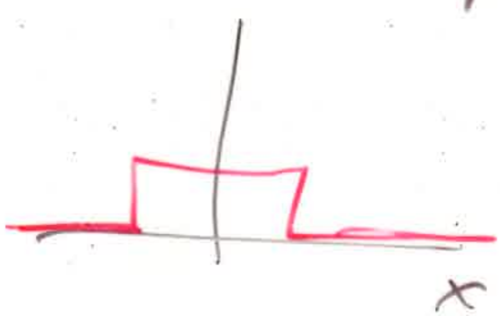
Gaussian $e^{-\frac{x^2}{2\sigma^2}}$ is its own Fourier transform

$\Delta x = \sigma_x$

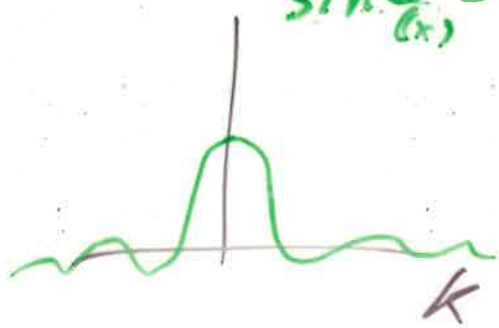


$\Delta k = \sigma_k$

$$\Delta x \cdot \underbrace{\Delta k \hbar}_{\Delta p} = \frac{\hbar}{2}$$



$\text{sinc}(x) = \frac{\sin(x)}{x}$



$$\Delta x \cdot \underbrace{\Delta k \hbar}_{\Delta p} \geq \frac{\hbar}{2}$$

The plane waves e^{+ikx}, e^{-ikx} are complete

= can expand any function in those basis vectors

complex numbers

basis vectors

define $$\Psi(x,t) = \sum_{n=0}^{\infty} c_n f_n(x) \cdot g_n(t)$$

$$c_n = \langle f_n \mid \psi(x) \rangle$$

$$e^{-\frac{itE}{\hbar}}$$

now $$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{k=-\infty}^{+\infty} c(k)\, e^{ikx} e^{-i\omega t}\, dk$$

basis vectors

complex function

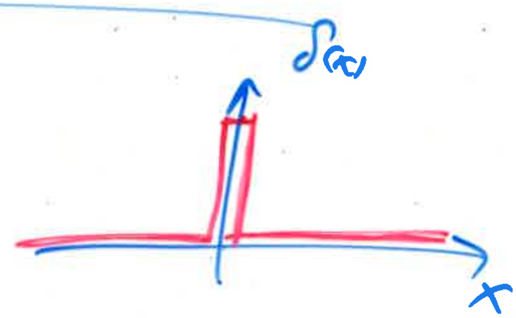
$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{k=-\infty}^{+\infty} c(k)\, e^{ikx}\, dk$$

$$c(k) = \langle e^{ikx} \mid \Psi(x,0) \rangle = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-ikx}\, \Psi(x,0)\, dx$$

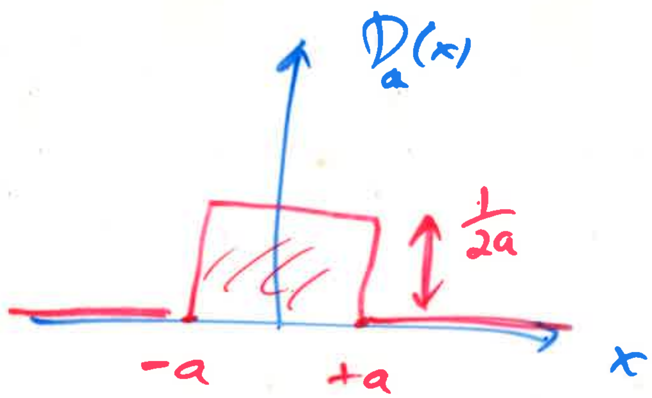
$$\langle e^{ikx} \mid e^{inx} \rangle = \begin{cases} 0, & k \neq n \\ 2\pi\,\delta(k-n), & k = n \end{cases}$$

Dirac Delta Function Potential

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$



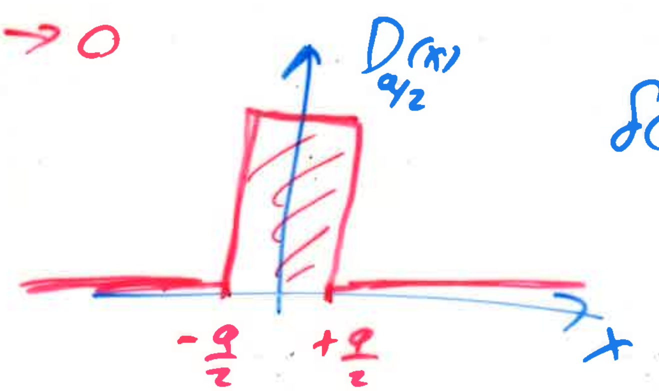
Can think of $\delta(x)$ as the limit of $D(x)$



$$D_a(x) = \begin{cases} 0 & , |x| > a \\ \frac{1}{2a} & , |x| < a \end{cases}$$

$$\int_{x=-\infty}^{+\infty} D_a(x) = 1$$

Limit $a \to 0$



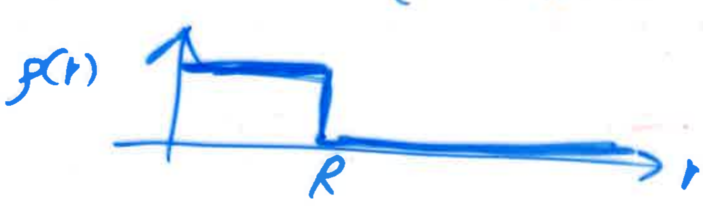
$$\delta(x) = \lim_{a \to 0} D_a(x)$$

Charge Density

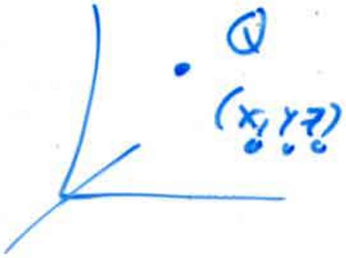
solid sphere

Q, R

$$\rho(r) = \begin{cases} \frac{Q}{\frac{4}{3}\pi R^3} & , r < R \\ 0, & r > R \end{cases}$$



Charge Density for a point charge



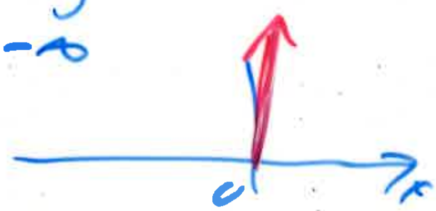
$$\rho(r) = Q\,\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$\int_{-\infty}^{+\infty}\!\!\int\!\!\int_{x\;y\;z} \rho(r)\,dx\,dy\,dz = Q$$

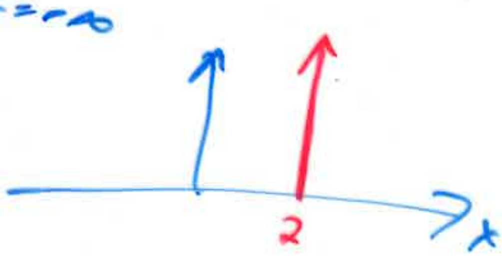
$$\int_{x=-\infty}^{\infty} \delta(x-x_0)\,dx = 1 \qquad \Bigg| \qquad \int_{y=-\infty}^{\infty} \delta(y-y_0)\,dy = 1$$

$$\int_{x=-\infty}^{+\infty} \delta(x)\,dx = 1$$

$\delta(x)$ is "piled up" wherever the argument is zero.



$$\int_{x=-\infty}^{+\infty} \delta(x-2)\,dx = 1$$



$$\int_{x=3}^{\infty} \delta(x-2)\,dx = 0$$

missed the pile-up.

$$\int_{x=-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)$$

Dirac delta

$$\int_{x=-\infty}^{\infty} \delta(x-2)\, f(x)\, dx = f(2)$$

Like: $\sum_{n=-\infty}^{\infty} \delta_{n2} f_n = f_2$

Kronecker delta

$$\int_{x=-\infty}^{\infty} \delta(\overset{a}{2x})\, f(x)\, dx$$

Change variable

$2x = y$ | $dx = \frac{dy}{2}$

$x = \frac{y}{2}$

$x = \pm\infty \Leftrightarrow y = \pm\infty$

$$= \int_{y=-\infty}^{\infty} \delta(y)\, f\left(\frac{y}{2}\right) \frac{dy}{2} = \underset{\uparrow a}{\frac{1}{2}} f(0)$$

$$\int_{-\infty}^{\infty} \delta(2x-7)\, f(x)\, dx$$

$y = 2x-7$

$dy = 2dx$

$$\int_{-\infty}^{\infty} \delta(y)\, f\left(\frac{y+7}{2}\right) \frac{dy}{2} = \frac{1}{2} f\left(\frac{7}{2}\right)$$