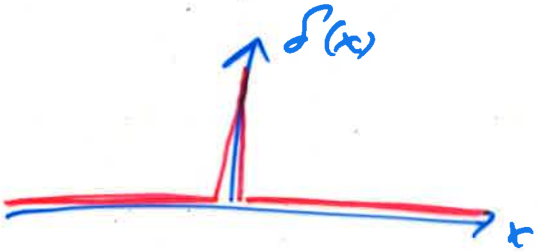
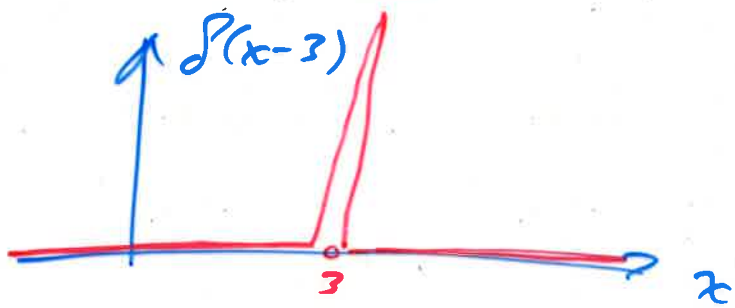
More notes on Dirac delta "function"

$x \Leftrightarrow k$
$t \Leftrightarrow \omega$





"Area" = 1 if you integrate over it.

$\delta(x)$ is not a function in the mathematical sense.

Because for a function $g(x)$

$$\lim_{\epsilon \to 0} \int_{x-\epsilon}^{x+\epsilon} g(x')\, dx' = 0$$

but $\displaystyle \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \delta(x)\, dx = 1$

$\delta(x)$ and others are called "generalized functions" or "distributions".

Must always appear under an integral sign.

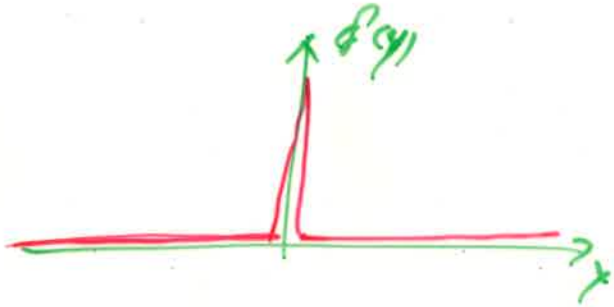
$\delta(x)$ is so singular that $[\delta(x)]^2$ does not make sense even under an integral.

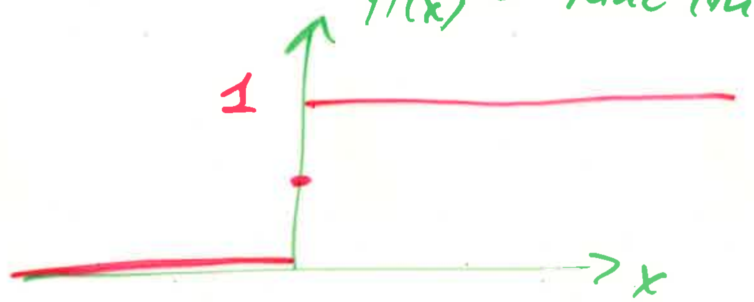
Integral of Dirac delta $\delta(x)$

ye

Oliver Heaviside

$$\theta(x) = H(x) = \int_{y=-\infty}^{x} \delta(y)\, dy$$





$H(0) = ? = 0, 1, \frac{1}{2}$ (Fourier series), $\frac{37}{\pi}$, $\sqrt{172}$ . . .

Derivative of Dirac delta $\dot{\delta}(x) \equiv \frac{d\delta(x)}{dx}$

$$\int_{x=-\infty}^{+\infty} \dot{\delta}(x)\, \varphi(x)\, dx$$

test function $\varphi(x)$ has bounded support

support = where $\varphi(x) \neq 0$

Integration by parts

$$d(uv) = (du)v + u(dv)$$

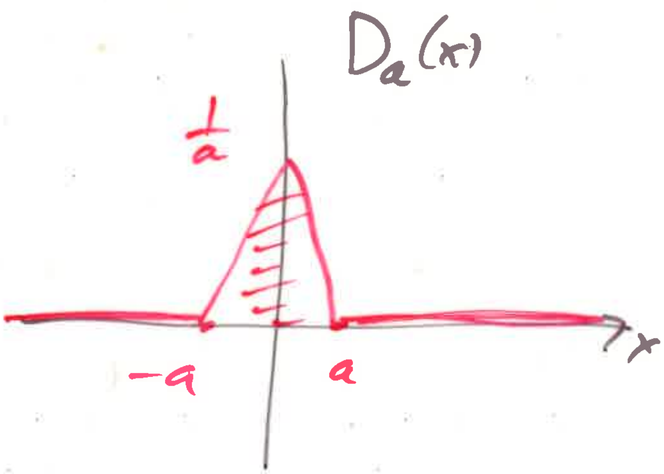
$$\int_a^b d(uv) = uv\Big|_a^b = \int_a^b v\, du + \int_a^b u\, dv$$

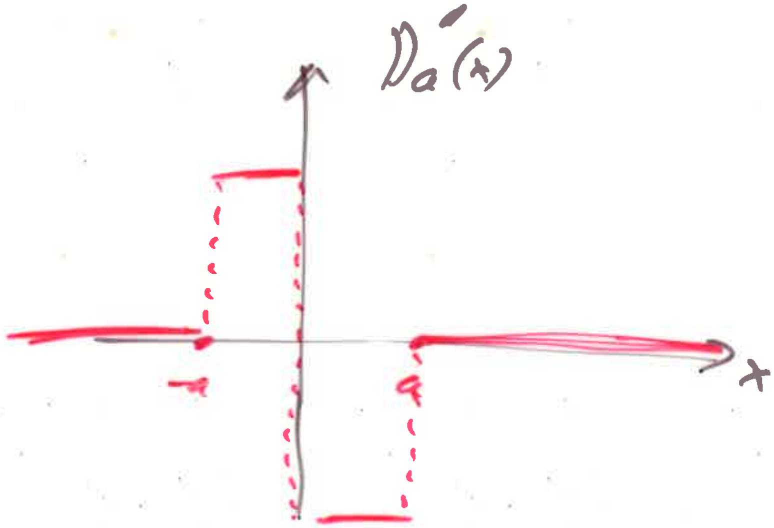
↑ surface term

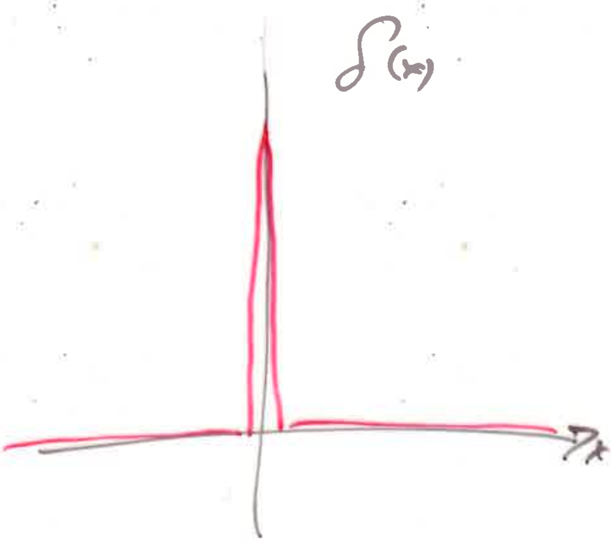
$$\int_a^b u\,dv = uv\Big|_a^b - \int v\,du$$

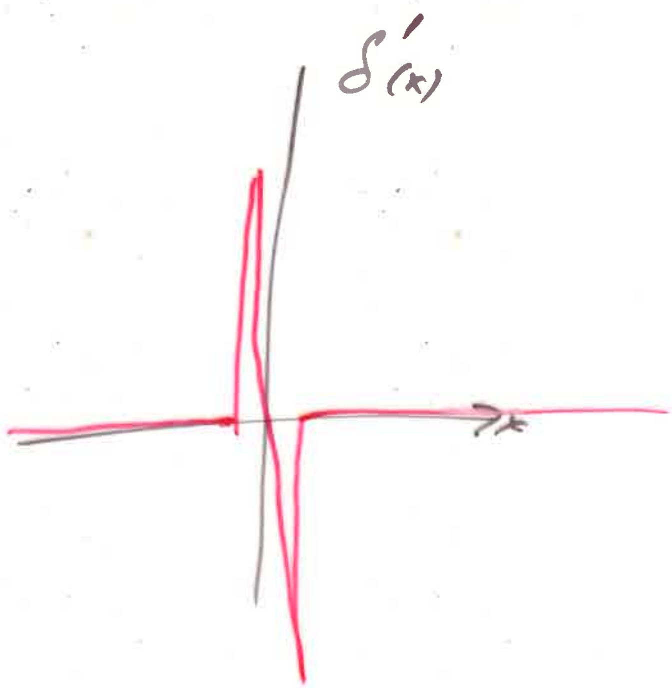
$$\int_{x=-\infty}^{\infty} \delta'(x)\,\varphi(x)\,dx = \overbrace{\cancel{\delta(x)\,\varphi(x)\Big|_{-\infty}^{\infty}}}^{\to 0} - \int_{-\infty}^{\infty} \delta(x)\,\varphi'(x)\,dx =$$

$$= -\varphi'(0) = -\frac{d\varphi(x)}{dx}\Bigg|_{x=0}$$









$$\underset{f(x)}{\Psi(x,0)} = \frac{1}{\sqrt{2\pi}} \int_{k=-\infty}^{+\infty} \underbrace{c(k)}_{\text{complex function}} \underbrace{e^{ikx}}_{\text{plane wave basis vectors}} dk$$

$$\vec{F} = \sum_{n=1}^{3} c_n \hat{u}_n$$

$$\psi(x) = \sum_{n=1}^{\infty} \underbrace{c_n}_{\text{complex coefficient}} \underbrace{f_n(x)}_{\text{basis vectors}}$$

$$\underset{\tilde{f}(k)}{c(k)} = \frac{1}{\sqrt{2\pi}} \int_{\cancel{x}\,y=-\infty}^{\infty} e^{-ik\cancel{x}\,y} \, \Psi(\cancel{x}\,y,0) \, d\cancel{x}\,y$$

$$c_n = \langle f_n | \psi(x) \rangle$$

$$c_n = \hat{u}_n \cdot \vec{F}$$

$c(k)$ is the (forward) Fourier transform of $\Psi(x,0)$

$\Psi(x,0)$ is the inverse Fourier transform of $c(k)$

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substitute $c(k)$ into top integral

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{k=-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{y=-\infty}^{\infty} e^{-iky} \Psi(y,0) \, dy \right] e^{ikx} \, dk$$

$$\Psi(x,0) = \int_{y=-\infty}^{\infty} \Psi(y,0) \underbrace{\left[\frac{1}{2\pi} \int_{k=-\infty}^{\infty} e^{ik(x-y)} \, dk \right]}_{\delta(x-y)} dy$$

$$f(x) = \int_{y=-\infty}^{\infty} f(y)\, \delta(x-y)\, dy$$

$$f_n = \sum_{w=1}^{\infty} f_w\, \delta_{nw}$$

$$\vec{f} = \underset{\approx}{\mathbb{I}} \cdot \vec{f}$$

↑ identity matrix

Fourier transform of Dirac delta

$$c_1(k) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} e^{-ik0} = \frac{1}{\sqrt{2\pi}}$$

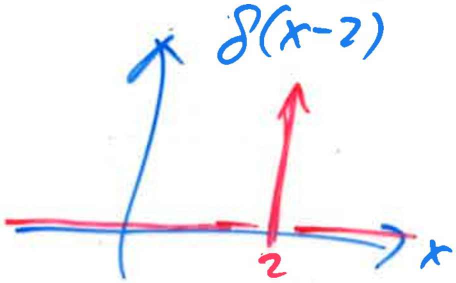
picked up at $x=0$

$$c_2(k) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} \delta(x-2)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} e^{-ik2}$$

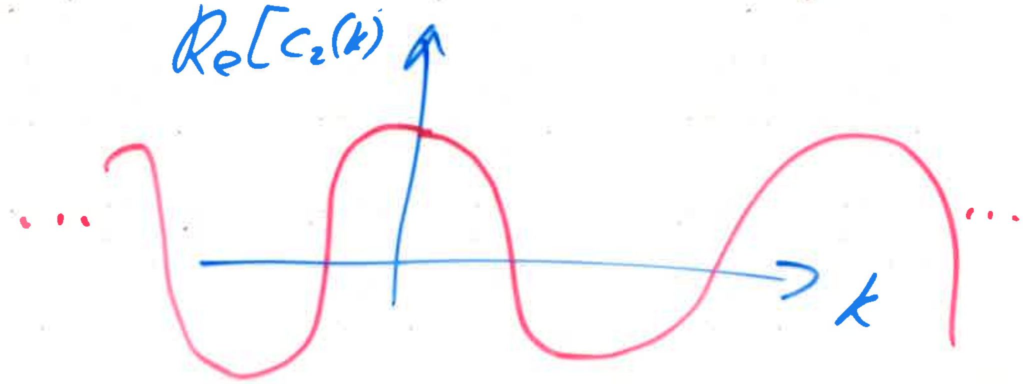
↑ Euler

$$= \frac{1}{\sqrt{2\pi}} \left[\cos(2k) - i \sin(2k) \right]$$

x - space
coordinate space



k - space $\quad p = \hbar k$
momentum space



Back to Physics

Dirac delta "function" potential $\quad V(x) = \pm \alpha\, \delta(x)$ $\quad \alpha > 0$

"strength" of potential

dimension of $\delta(x)$: $\displaystyle\int_{x=-\infty}^{\infty} \delta(x)\, dx = 1$

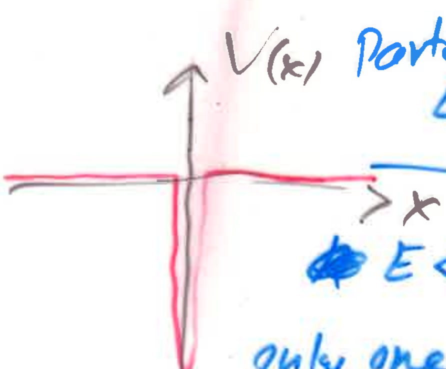
$\frac{1}{\text{Length}}$ $\quad$ Length $\quad$ dimensionless

MKS unit

$[\delta(x)] = \frac{1}{\text{meters}}$

Well

$V(x) = -\alpha\, \delta(x)$

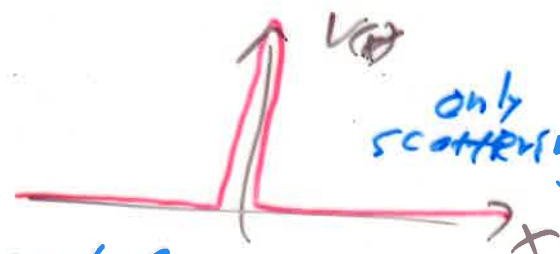


Particle energies
$E > 0$, any value
scattering state

$E < 0$, quantized
only one bound state

Barrier

$V(x) = \alpha\, \delta(x)$



only scattering

particle energies
$E > 0$ $\quad$ not quantized
any real value $E \propto k$

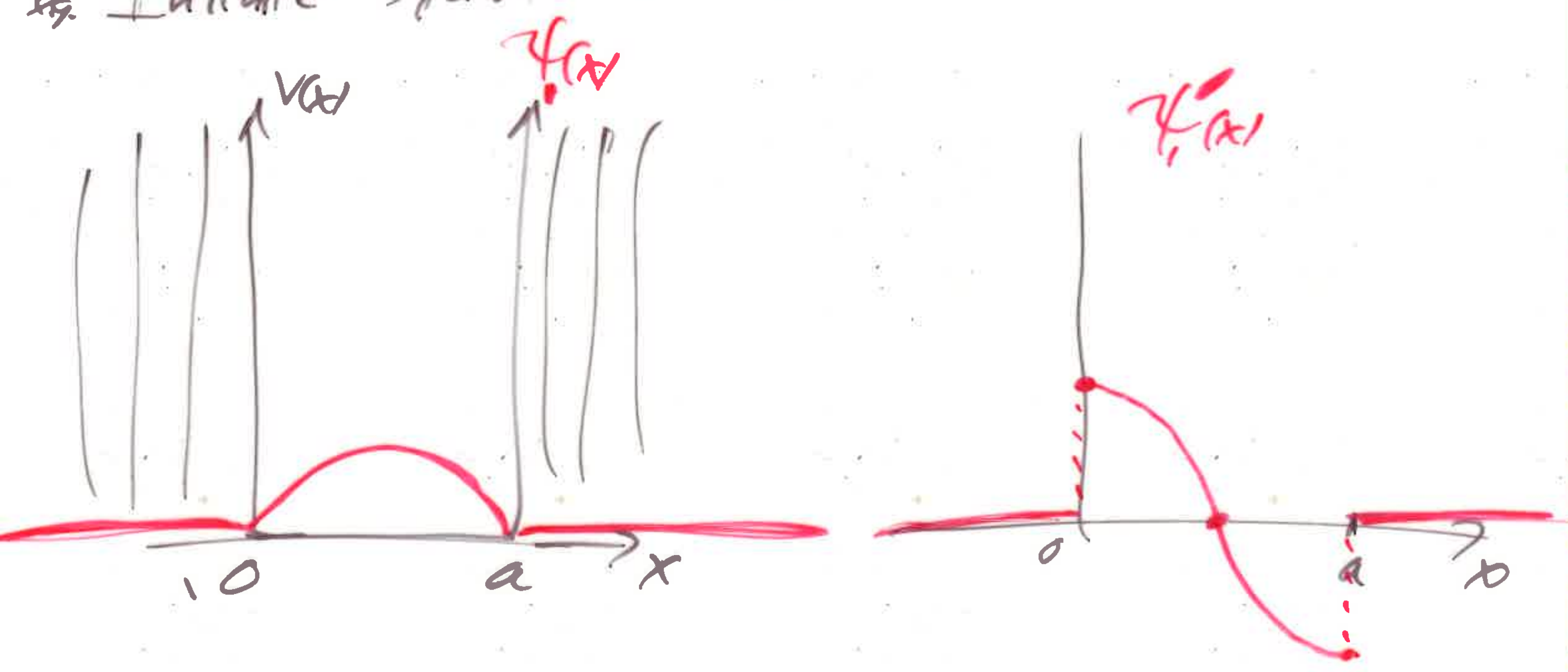
Delta function Well $V(x) = -\infty\,\delta(x)$

Solve to S.E.

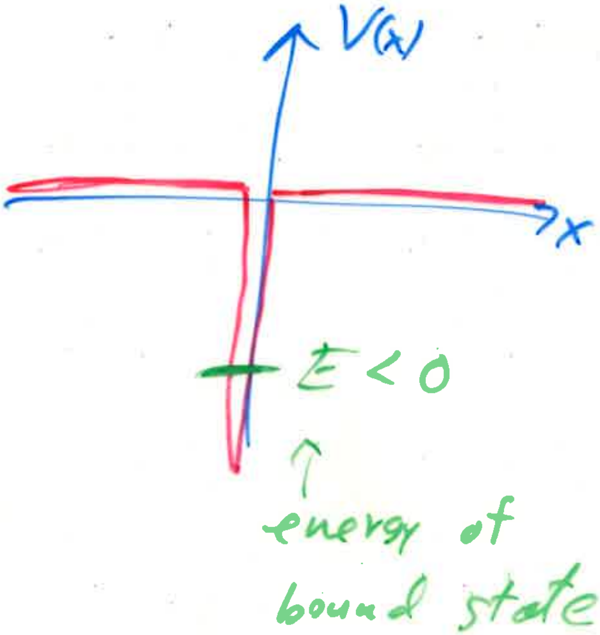
$$V(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ -\infty, & \text{if } x = 0 \end{cases}$$

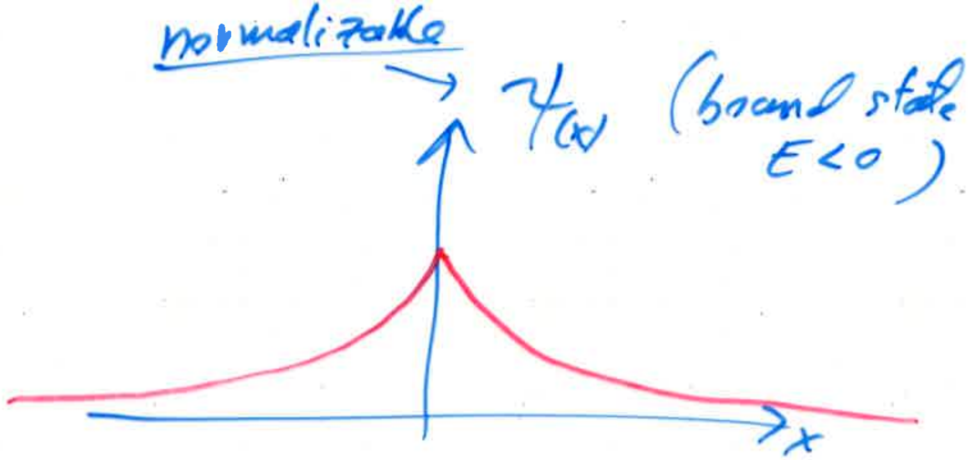
Plan: 1) Solve S.E. for $x<0$.
2) Solve S.E. for $x>0$.
3) Match the solutions (plane waves) at $x=0$
3a) $f(x) = \Psi(x,0)$ is continuous. ✓
3b) $f'(x) = \dfrac{d\Psi(x,0)}{dx}$ is normally continuous
except where $V(x)$ is infinite.

eg. Infinite Square Well Potential



Expectations:





$$\int_{x=-\infty}^{\infty} |\psi(x)|^2 dx = 1$$