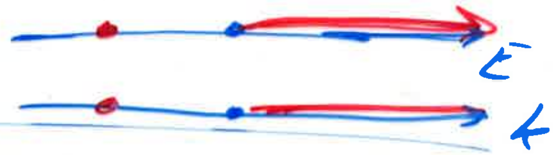
time solution $g(t) = e^{-\frac{itE}{\hbar}}$

Delta-function well



Bound states $E<0$

Scattering states $E>0$

S.E. $$-\frac{\hbar^2}{2m} f''(x) \underbrace{-\alpha\,\delta(x)}_{V(x)} f(x) = E f(x)$$

space part.

$x<0$ $$-\frac{\hbar^2}{2m} f_L''(x) = E f_L(x)$$ $E<0$

$$f_L''(x) \equiv \gamma^2 f_L(x)$$

kappa

$$\gamma = \sqrt{\frac{-2mE}{\hbar^2}} = \kappa$$

$\gamma \in$ Reals

$$f_L(x) = Ae^{-\gamma x} + Be^{\gamma x}$$ ← could also write $A' \cosh(\gamma x) + B' \sinh(\gamma x)$

When $x \to -\infty$, $e^{-\gamma x} \to \infty \Rightarrow A = 0$

$$f_L(x) = Be^{\gamma x}$$

$x>0$ $$-\frac{\hbar^2}{2m} f_R''(x) = E f_R(x)$$ $\boxed{E<0}$

$$f_R''(x) = \gamma^2 f_R(x) \Longrightarrow f_R(x) = Fe^{-\gamma x} + Ge^{\gamma x}$$

when $x \to +\infty$, $e^{\gamma x} \to \infty \Rightarrow G = 0$

$$f_R(x) = Fe^{-\gamma x}$$

$$f_L(x) = Be^{\gamma x} \qquad f_R(x) = Fe^{-\gamma x}$$

$f(x)$ is continuous across the boundaries.

$$f_L(0) = f_R(0) \Longrightarrow Be^{\gamma 0} = Fe^{-\gamma 0} \Longrightarrow B = F$$

$$f(x) = \begin{cases} Be^{\gamma x}, & x \le 0 \\ Be^{-\gamma x}, & x \ge 0 \end{cases}$$

Still 2 unknowns
B, γ.
Now what?

Integrate the S.E. over a little neighborhood around $x=0$:

$$\lim_{\epsilon \to 0} \int_{x=-\epsilon}^{+\epsilon} S.E.\, dx$$

$$\lim_{\epsilon \to 0} \left[-\frac{\hbar^2}{2m} \int_{x=-\epsilon}^{\epsilon} \frac{d^2 f(x)}{dx^2} dx - \alpha \int_{x=-\epsilon}^{\epsilon} \delta(x) f(x)\, dx = E \int_{x=-\epsilon}^{\epsilon} f(x)\, dx \right]$$

$$\lim_{\epsilon \to 0} -\frac{\hbar^2}{2m} \left. \frac{df(x)}{dx} \right|_{-\epsilon}^{\epsilon} - \alpha f(0) = 0$$

$$-\frac{\hbar^2}{2m} \left[f_R'(0) - f_L'(0) \right] = \alpha f(0)$$

L or R

$$f_L'(x) = \gamma B e^{\gamma x}$$

$$f_R'(x) = -\gamma B e^{-\gamma x}$$

$$-\frac{\hbar^2}{2m}\left[(-\gamma B) - (\gamma B)\right] = \alpha B$$

$\gamma > 0$

$$\frac{\hbar^2}{2m} 2\gamma \cancel{B} = \alpha \cancel{B} \implies \gamma = \frac{m\alpha}{\hbar^2} = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$E = -\frac{m\alpha^2}{2\hbar^2} < 0$$

one bound state.

$$f(x) = \begin{cases} B e^{\frac{m\alpha}{\hbar^2}x}, & x \le 0 \\ B e^{-\frac{m\alpha}{\hbar^2}x}, & x \ge 0 \end{cases} \qquad B = ?$$

$$1 = \int_{x=-\infty}^{+\infty} |\psi|^2 dx = \int_{x=-\infty}^{+\infty} |f(x)|^2 dx = |B|^2 \int_{x=-\infty}^{+\infty} \cdots$$

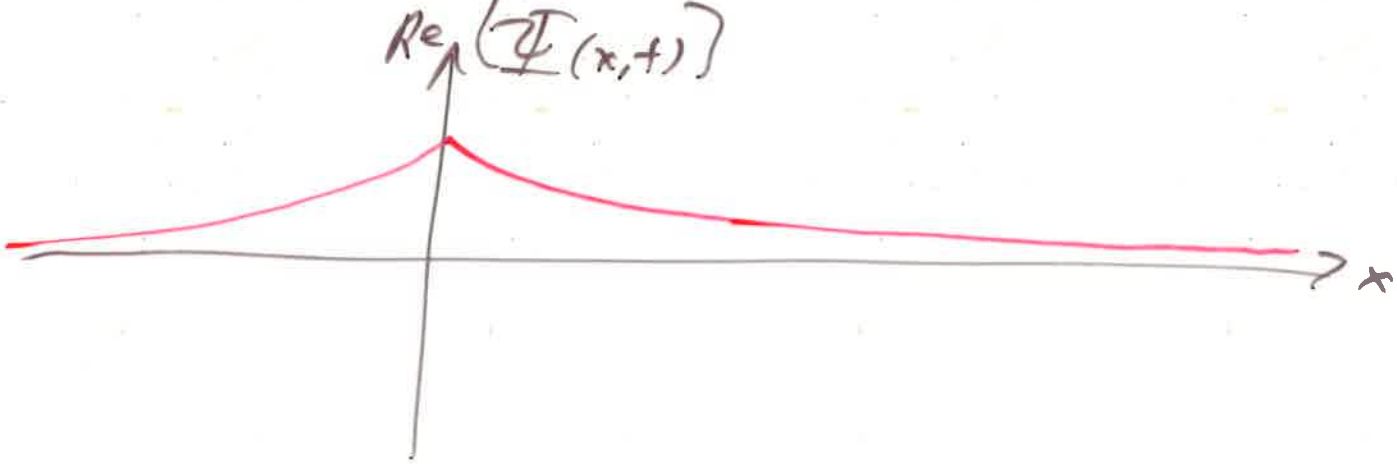
$$1 = |B|^2 \left[\int_{x=-\infty}^{0} e^{\frac{2m\alpha}{\hbar^2}x} dx + \int_{x=0}^{+\infty} e^{-\frac{2m\alpha}{\hbar^2}x} dx\right]$$

$$1 = |B|^2 \cancel{2} \frac{1}{\left(-\frac{\cancel{2}m\alpha}{\hbar^2}\right)} e^{-\frac{2m\alpha}{\hbar^2}x}\Big|_{x=0}^{\infty} = -|B|^2 \frac{\hbar^2}{m\alpha}\left[0 - 1\right]$$

$$|B|^2 = \frac{m\alpha}{\hbar^2} \Rightarrow B = \frac{\sqrt{m\alpha}}{\hbar} \quad \cancel{e^{i\theta}}$$

$$\Psi(x,t) = f(x)\cdot g(t) = B e^{-\frac{m\alpha}{\hbar^2}|x|} e^{-\frac{itE}{\hbar}}$$

$$= \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha}{\hbar^2}|x|} e^{\frac{it}{\hbar}\cdot\frac{m\alpha^2}{2\hbar^2}}$$



Delta Function Well – Scattering States $E>0$

Any E at all
no quantization

$x<0$ S.E. $-\frac{\hbar^2}{2m} f_L''(x) = E f_L(x)$ $\qquad E>0$

$$f_L''(x) = -k^2 f_L(x) \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

wavenumber.

$$f_L(x) = A e^{ikx} + B e^{-ikx}$$

$A' \sin(kx) + B' \cos(kx)$

$x>0$ S.E. $-\frac{\hbar^2}{2m} f_R''(x) = E f_R(x)$ $\qquad E>0$

$$f_R(x) = F e^{ikx} + G e^{-ikx}$$

$i(kx-\omega t) \longrightarrow$

$$g(t) = e^{-\frac{itE}{\hbar}} = e^{-i\omega t}$$

$\omega = \frac{E}{\hbar} \Rightarrow \hbar\omega = E$

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1) Continuity of $f(x)$ across the boundary $x=0$

$$f_L(0) = f_R(0) \Longrightarrow \boxed{A+B = F+G}$$

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2) Derivatives

$$f_L'(x) = \frac{df_L}{dx} = ik\left(Ae^{ikx} - Be^{-ikx}\right)$$

$$f_R'(x) = \frac{df_R}{dx} = ik\left(Fe^{ikx} - Ge^{-ikx}\right)$$

Discontinuity of $f'(x)$ at boundary $x=0$

$$-\frac{\hbar^2}{2m}\left[f_R'(0) - f_L'(0)\right] = \alpha f(0)$$

look

$$\boxed{-\frac{\hbar^2 ik}{2m}\left[F - G - (A-B)\right] = \alpha\,(A+B)}$$

OK $(F+G)$

2 Equations - 4 unknowns! Now what?

$\overrightarrow{Ae^{ikx}}$ incoming wave

$\overleftarrow{Be^{-ikx}}$ reflected wave

$\overrightarrow{Fe^{ikx}}$ transmitted wave

$G=0$ no incoming waves from the right.

① $A+B = F$

② $ik(F-A+B) = -\dfrac{2m\alpha}{\hbar^2}F$

$\Big\}$ 3 unknowns, 2 equations