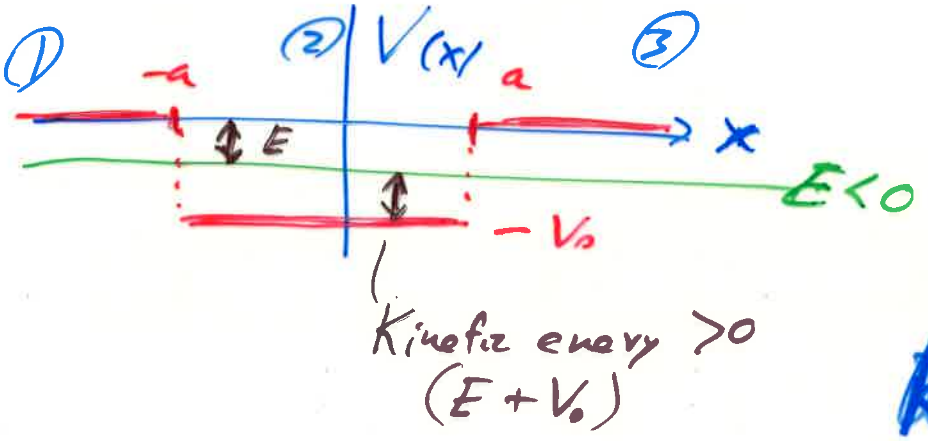


① $x < -a$

② $-a < x < a$

③ $a < x$

$$k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

Look for <u>odd</u> bound state solutions

$$f(x) = -f(-x)$$

$$\gamma = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$f_1(x) = B e^{\gamma x} \cancel{+ A e^{-\gamma x}}$$

$$f_1'(x) = B\gamma e^{\gamma x}$$

$$f_2(x) = C \sin(kx)$$

$$f_2'(x) = Ck\cos(kx)$$

$$f_3(x) = F e^{-\gamma x} \cancel{+ G e^{\gamma x}}$$

$$f_3'(x) = -\gamma F e^{-\gamma x}$$

Match $f(x)$ and $f'(x)$ at boundaries $x = \pm a$

① $f_1(-a) = f_2(-a) \Rightarrow B e^{-\gamma a} = -C\sin(ka)$

② $f_2(a) = f_3(a) \Rightarrow C\sin(ka) = F e^{-\gamma a}$

$\Big\}\ B = -F$

---

③ $f_1'(-a) = f_2'(-a) \Rightarrow B\gamma e^{-\gamma a} = Ck\cos(ka)$

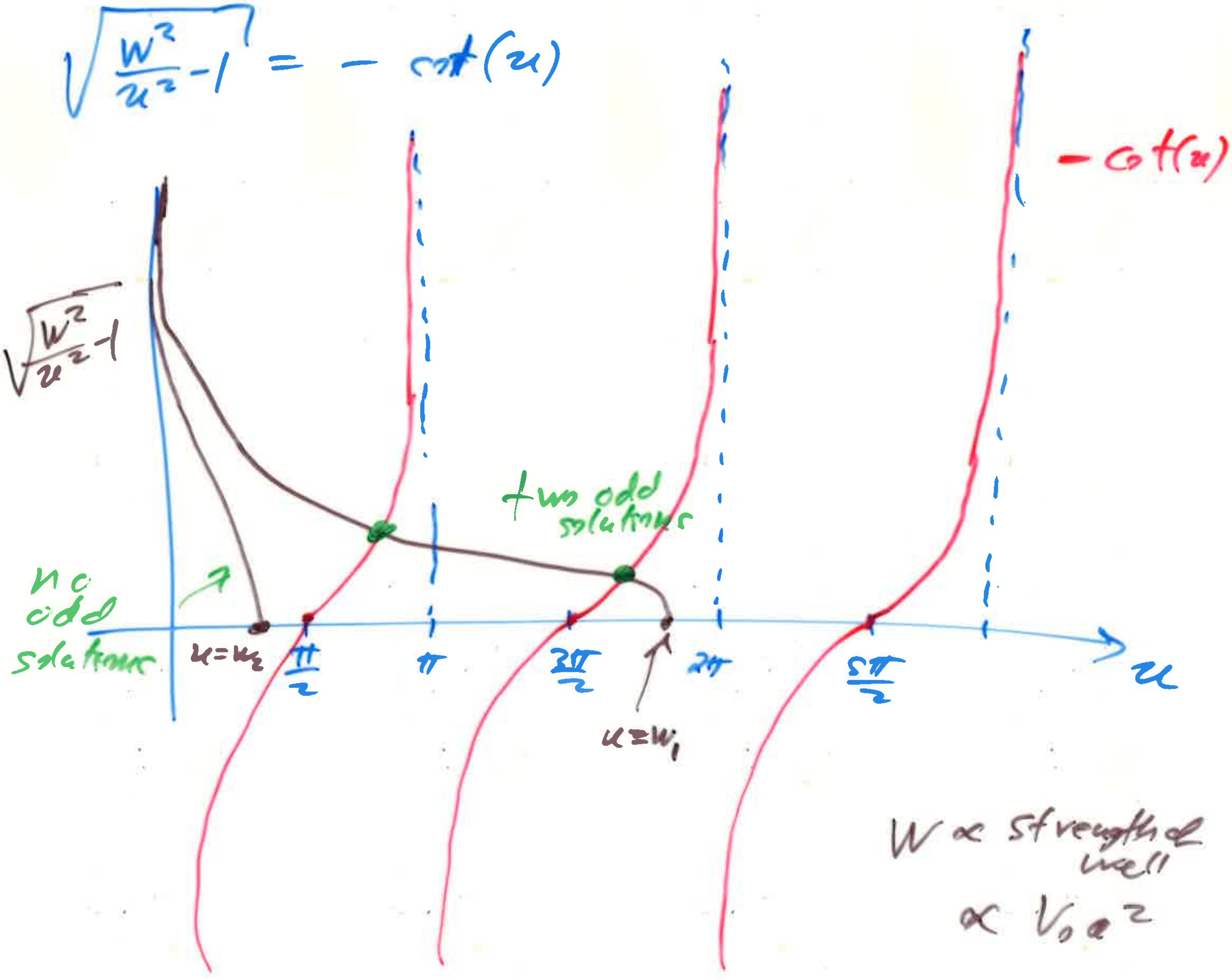
④ $f_2'(a) = f_3'(a) \Rightarrow Ck\cos(ka) = -\gamma F e^{-\gamma a}$

$\Big\}\ B = -F$

$$\frac{③\ \cancel{B}\gamma e^{\cancel{-\gamma a}} = \cancel{C} k\cos(ka)}{①\ \cancel{B} e^{\cancel{-\gamma a}} = -\cancel{C}\sin(ka)} \Rightarrow \gamma a = -ka\cot(ka)$$

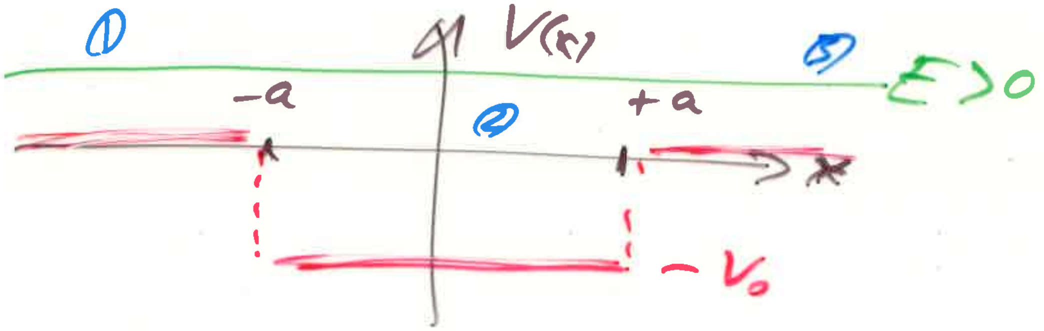
$$u \equiv ka\ ,\ w \equiv \frac{2mV_0a^2}{\hbar^2} \Longrightarrow \sqrt{w^2-u^2} = -u\cot(u)$$

$$\sqrt{\frac{w^2}{u^2}-1} = -\cot(u)$$



$W \propto$ strength of well $\propto V_0a^2$

Finite Square Well - Scattering States $E > 0$



E isn't quantized

$$f_1(x) = \underset{\text{Inc}\rightarrow}{A e^{-ikx}} + \underset{\leftarrow\text{refl.}}{B e^{ikx}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \leftarrow K.E$$

$$f_2(x) = \underset{\rightarrow}{C e^{-i\delta x}} + \underset{\leftarrow}{D e^{i\delta x}}$$

$$\delta = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

no incoming waves from $x = +\infty$

$$f_3(x) = \underset{\text{trans.}\rightarrow}{F e^{-ikx}} + \underset{\leftarrow}{\cancelto{0}{G} e^{ikx}}$$

$\sqrt{2m(K.E)}$ = momentum

---

4 equations $f(x)$, $f'(x)$ are continuous across any boundaries $x = \pm a$

$$f_1(-a) = f_2(-a) \quad \Big| \quad f_1'(-a) = f_2'(-a)$$

$$f_2(a) = f_3(a) \quad \Big| \quad f_2'(a) = f_3'(a)$$

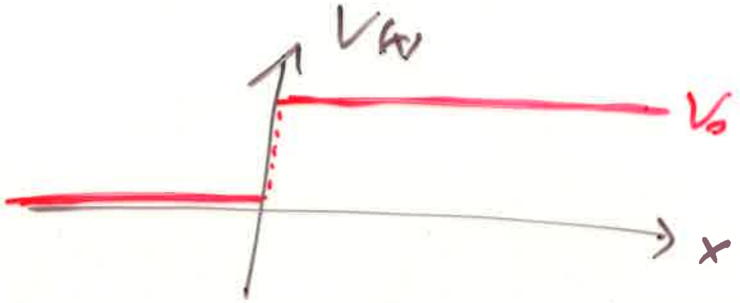
unknowns $A, B, C, D, F, \cancelto{0}{G}$

divide through by A

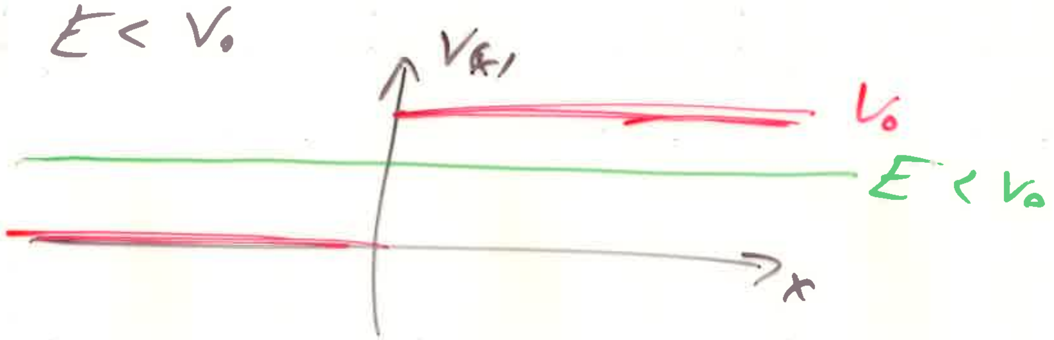
$$\frac{B}{A} = \tilde{B}, \quad \frac{C}{A} = \tilde{C} \; \ldots, \; \tilde{D}, \; \tilde{F}$$

Other Potentials

① Potential Step

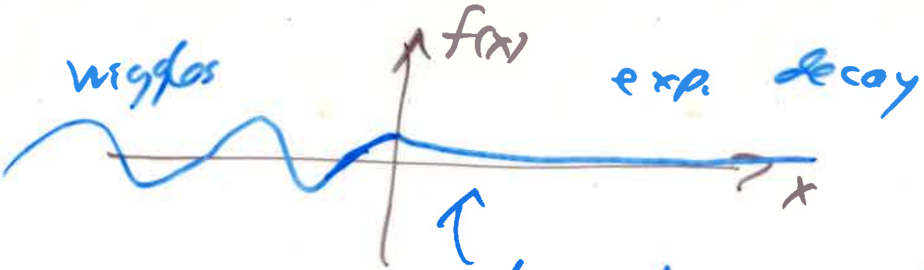


1a) $E < V_0$



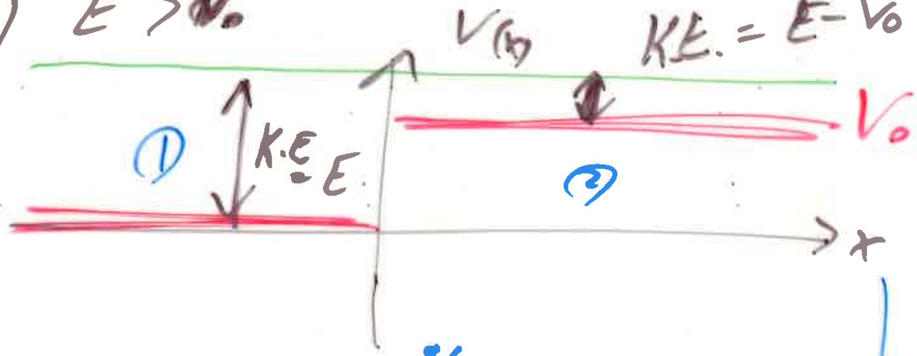
$R = 1$

$T = 0$



tunneling into the classically forbidden region

1b) $E > V_0$



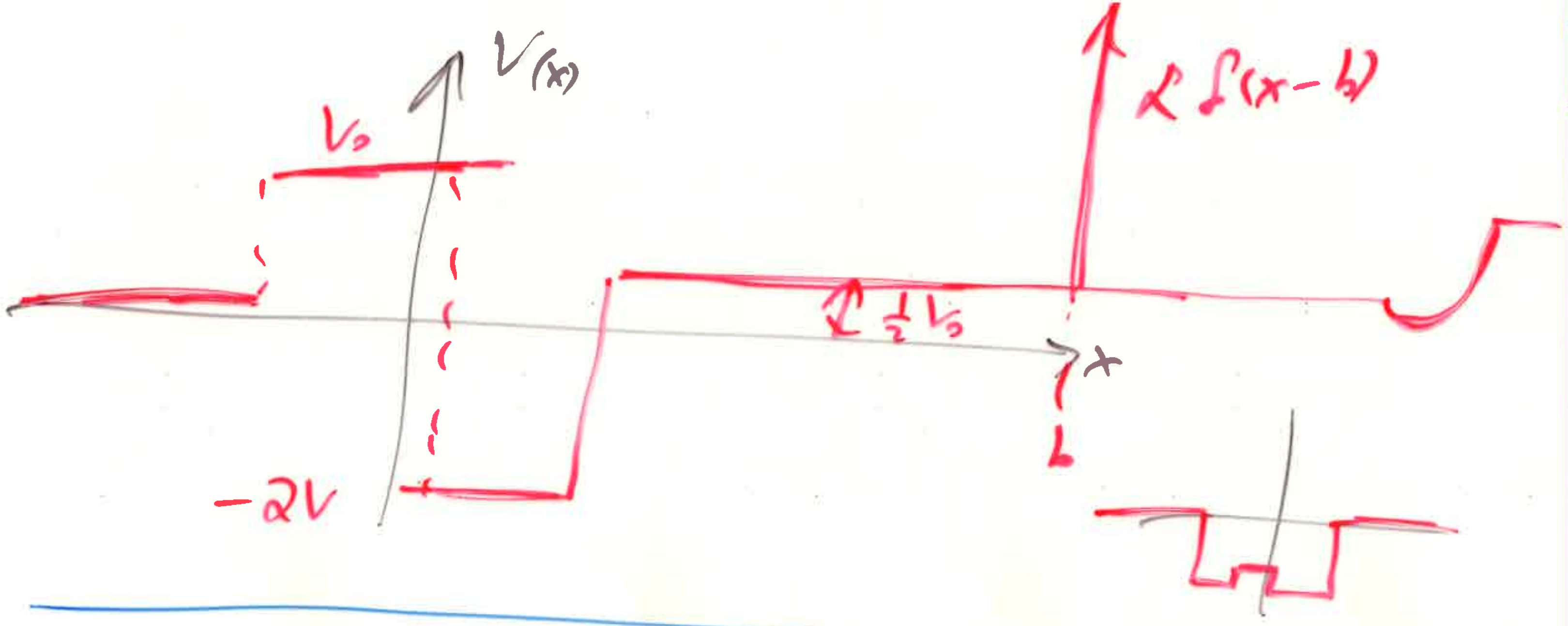
$$f_1(x) = \underset{\overrightarrow{\text{inc}}}{A e^{-ikx}} + \underset{\longleftarrow}{B e^{ikx}} \quad \text{refl}$$

$$f_2(x) \quad \underset{\overrightarrow{\text{trans}}}{F e^{-ikx}} + \underset{\longleftarrow}{\overset{\nearrow 0}{\cancel{B}}\, e^{ikx}}$$

$$R = \left|\frac{B}{A}\right|^2 = \tilde{B}^* \tilde{B}$$

$$T = \frac{v_t}{v_i}\left|\frac{F}{A}\right|^2 = \tilde{F}^* \tilde{F} \frac{v_t}{v_i}$$

$$= \sqrt{\frac{E - V_0}{E}}\, \tilde{F}^* \tilde{F}$$



Numerical solutions to the S.E.

$$-\frac{\hbar^2}{2m}\frac{d^2 f(x)}{dx^2} + \underbrace{\frac{1}{2}m\omega^2x^2}_{V(x)} f(x) = E f(x)$$

Problems $[V(x)] = [E] = \frac{L^2 M}{T^2}$, $[x] = L$, $[m] = M$

Everything must be a pure dimensionless number for a computer.

Define $u = \frac{x}{l_0}$ where l_0 is a constant length.

$$x = l_0 u \qquad dx = l_0\, du$$

SE: $-\frac{\hbar^2}{2m}\frac{d^2 f(u)}{l_0^2\, du^2} + \frac{1}{2}m\omega^2 l_0^2 u^2 f(u) = E f(u)$

$$f''(u) - \frac{m^2\omega^2 l_0^4}{\hbar^2} u^2 f(u) = -\frac{2m l_0^2 E}{\hbar^2} f(u)$$

Define: $\ell_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$

$$f''(u) - u^2 f(u) = \left(\frac{-E}{(\frac{1}{2}\hbar\omega)}\right) f(u) = -\epsilon f(u)$$

$$\epsilon = \frac{E}{\frac{1}{2}\hbar\omega}$$

$$f''(u) - (u^2 - \epsilon) f(u) = 0$$

Next time - First-order Euler method.