$\underline{\underline{S}}^T = \underline{\underline{S}}$ symmetric

$\underline{\underline{A}}^T = -\underline{\underline{A}}$ antisymmetric

$\underline{\underline{O}}^T = \underline{\underline{O}}^{-1}$ orthogonal $\quad \underline{\underline{O}}^T \underline{\underline{O}} = \underline{\underline{\mathbb{I}}}$ identity matrix

$\underline{\underline{R}}^* = \underline{\underline{R}}$ real

$\underline{\underline{H}}^\dagger = H$ Hermitian $\quad$ ($\dagger$ dagger $= \underset{\text{transpose}}{T} \overset{\text{complex conjugate}}{*}$)

$U^\dagger = U^{-1}$ unitary. $\quad \underline{\underline{U}}^\dagger \underline{\underline{U}} = \underline{\underline{\mathbb{I}}}$

$$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$$

$$\det(\mathbb{I}_{n\times n}) = +1 \longleftrightarrow \det(-\mathbb{I}_{n\times n}) = (-1)^n$$

$$\det(\underline{\underline{A}}^{-1}) = \frac{1}{\det(\underline{\underline{A}})}$$

$$\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \cdot \det(\underline{\underline{B}})$$

Quantum Mechanical Observables (Operator)

$\Longleftrightarrow$ Hermitian Matrices

A matrix $\underline{\underline{H}}$ is Hermitian if $\underline{\underline{H}} = \underline{\underline{H}}^{\dagger}$ ← dagger

$\dagger = T\,*$ transpose, complex conjugate

The eigenvalues of a Hermitian matrix are real.

① $\langle v|\hat{H}|v\rangle = \lambda\langle v|v\rangle$

② $\langle v|\hat{H}^{\dagger}|v\rangle = \lambda^{*}\langle v|v\rangle$

$\quad\;\; \|$

$\langle v|\hat{H}|v\rangle$

$\underline{\underline{H}}\vec{v} = \lambda\vec{v}$

$\vec{v}^{\dagger}H^{\dagger} = \lambda^{*}\vec{v}^{\dagger}$

$\vec{v}^{\dagger}\cdot(\underline{\underline{H}}\vec{v}) = \lambda(\vec{v}^{\dagger}\cdot\vec{v})$

Subtract $0 = (\lambda-\lambda^{*})\langle v|v\rangle$

$\langle v|v\rangle = 0$ or $\lambda = \lambda^{*} \Longrightarrow \lambda \in \text{Real}$ ✓

$\hookrightarrow |v\rangle = 0$

0 is not a eigenvector

Eigenvectors corresponding to <u>different</u> eigenvalues of a Hermitian matrix are orthogonal.

$$\hat{H}|v\rangle = \lambda\, |v\rangle \qquad \langle v|\hat{H}^{+} = \lambda^{*}\langle v|$$

① $\langle v|\hat{H}\,|u\rangle = \rho\langle v|u\rangle$ ② $\langle v|\hat{H}|u\rangle = \lambda^{*}\langle v|u\rangle = \lambda\,\langle v|u\rangle$

Subtract ①-② $\Longrightarrow$ $0 = (\rho-\lambda)\,\langle v|u\rangle$

$\cancel{\rho=\lambda}$ <u>or</u> $\langle v|u\rangle = 0$ ✓

If two eigenvalues are the same

$\lambda_1 = \lambda_2 \neq \lambda_{3,4,5}$ —

The two corresponding eigenvectors $|v_1\rangle, |v_2\rangle$ form an eigen subspace

e.g. $|W\rangle = \alpha\,|v_1\rangle + \beta\,|v_2\rangle$

$$H|w\rangle = \lambda_1\,|w\rangle = \lambda_2\,|w\rangle$$

We can always <u>choose</u> a set of orthogonal vectors that span the eigen subspace.

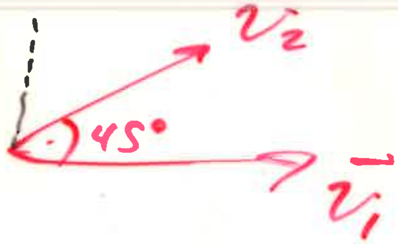
$$\hat{H} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$\lambda_1 = 2 \qquad |v_1\rangle = (1,0,0)$

$\lambda_2 = 2 \qquad |v_2\rangle = (0,1,0)$

$\boxed{\lambda_3 = 5 \qquad |v_3\rangle = (0,0,1)}$

$|v_1\rangle = (1,0,0) \qquad |v_2'\rangle = \frac{1}{\sqrt{2}}(1,1,0)$



$|v_2'\rangle = \frac{1}{\sqrt{2}}(1,1,0)$

$|v_1'\rangle = \frac{1}{\sqrt{2}}(1,-1,0)$

$|u_1\rangle = (\sin\theta, \cos\theta, 0) \qquad \hat{H}|u_1\rangle = 2|u_1\rangle$

$|u_2\rangle = (-\cos\theta, \sin\theta, 0) \qquad \hat{H}|u_2\rangle = 2|u_2\rangle$

Spectral Decomposition of Hermitian Matrices

If $\underline{\underline{H}}|v_i\rangle = \lambda_i|v_i\rangle$ then

the matrix $\underline{\underline{P}}_i = |v_i\rangle\langle v_i|$ is projector onto the ith eigenspace

$$\underline{\underline{P}}_i \, \underline{\underline{P}}_k = \delta_{ik} \, \underline{\underline{P}}_k$$

$$\underline{\underline{H}} = \sum_i \lambda_i \, \underline{\underline{P}}_i$$ is the spectral decomposition.
(expansion in eigenvector basis)

$$\underline{\underline{M}} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} = \underline{\underline{M}}^{\dagger} \quad \checkmark \quad \text{Hermitian}$$

$$\underline{\underline{M}}\,|v_1\rangle = \lambda_1 |v_1\rangle \qquad \lambda_1 = 4, \quad |v_1\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

normalized

$$\langle v_1 | v_1 \rangle = 1^2 + (-2)^2 = 5$$

$$\underline{\underline{M}}\,|v_2\rangle = \lambda_2 |v_2\rangle \Rightarrow \lambda_2 = 9 \qquad |v_2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Form Projector — Dyad

$$\underline{\underline{P_1}} = |v_1\rangle\langle v_1| = \vec{v}_1 \, \vec{v}_1^{\,\dagger} =$$

$$\begin{pmatrix} \vdots \end{pmatrix} (\cdots)$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\underline{\underline{P_2}} = |v_2\rangle\langle v_2| = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{5}} (2 \;\; 1)$$

$$= \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\underline{\underline{M}} = \sum_{i=1}^{2} \lambda_i \underline{\underline{P_i}} = \lambda_1 \underline{\underline{P_1}} + \lambda_2 \underline{\underline{P_2}}$$

$$\underline{\underline{M}} = \frac{4}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} + \frac{9}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\underline{\underline{M}} = \sum_i \lambda_i \underline{\underline{P}}_i$$

$$\underline{\underline{M}}^2 = \left(\sum_i \lambda_i \underline{\underline{P}}_i\right)\left(\sum_k \lambda_k \underline{\underline{P}}_k\right)$$

$$= \sum_i \sum_k \lambda_i \lambda_k \underline{\underline{P}}_i \underline{\underline{P}}_k = \sum_i \cancel{\sum_k} \lambda_i \underset{\uparrow i}{\lambda_k} \overset{1}{\cancel{\delta_{ik}}} \underline{\underline{P}}_i$$

$$= \sum_i (\lambda_i)^2 \underline{\underline{P}}_i$$

$$\underline{\underline{M}}^n = \sum_i (\lambda_i)^n \underline{\underline{P}}_i$$

$$e^{\underline{\underline{M}}} \equiv \underline{\underline{I}} + \underline{\underline{M}} + \frac{1}{2!}\underline{\underline{M}}^2 + \frac{1}{3!}\underline{\underline{M}}^3 + \cdots$$

$$e^{M} = \sum_{i=1}^{2} \left(e^{\lambda_i}\right) \underline{\underline{P}}_i$$

$$e^{\begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}} = \frac{e^4}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} + \frac{e^9}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^4 + 4e^9}{5} & -\frac{2}{5}e^4 + \frac{2}{5}e^9 \\ -\frac{2}{5}e^4 + \frac{2}{5}e^9 & \frac{4}{5}e^4 + \frac{1}{5}e^9 \end{pmatrix}$$

$$R = \sqrt{M} = \sqrt{\begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}} \qquad R^2 \stackrel{?}{=} M$$

$$= \sum_{i=1}^{2} \sqrt{\lambda_i}\, \underline{\underline{P_i}} = \frac{\pm 2}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} + \frac{\pm 3}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

4 Matrix roots R_1, R_2, R_3, R_4

$$f(\underline{\underline{M}}) = \sum_i f(\lambda_i)\, \underline{\underline{P_i}}$$