Last time if $\hat{H} = \hat{H}^\dagger$ (hermitian)

eigenvalues + vectors $\hat{H}|v_i\rangle = \lambda_i |v_i\rangle$

operator (matrix) ↗, ket vector ↑, real ~~complex~~ number ↖↗

Projector

$$\hat{P}_i = |v_i\rangle\langle v_i|$$

outer product

Dot product

$$\langle v_i | v_i \rangle = 1$$

inner product
scalar product

Spectral Decomposition

$$\hat{H} = \sum_i \lambda_i |v_i\rangle\langle v_i| = \sum_i \lambda_i \hat{P}_i$$

$$f(\hat{H}) = \sum_i f(\lambda_i) |v_i\rangle\langle v_i| = \sum_i f(\lambda_i) \hat{P}_i$$

e.g. $\sin(\hat{H}) = \sum_i \sin(\lambda_i) \hat{P}_i$

Suppose $f(x) = x^0 = 1$

$$(\hat{H})^0 = \hat{\mathbb{I}} = \sum_i \lambda_i^0 \hat{P}_i = \sum_i \hat{P}_i = \sum_i |v_i\rangle\langle v_i|$$

↑ identity matrix

If continuous, not discrete

$$\mathbb{I} = \int_{-\infty}^{\infty} |k\rangle\langle k| \, dk$$

Can insert $\hat{\mathbb{I}}$ anywhere

$$\hat{A}|b\rangle = \hat{A}\hat{\mathbb{I}}|b\rangle = \hat{A}\left(\sum_i |v_i\rangle\langle v_i|\right)|b\rangle$$

$$= \sum_i \hat{A}|v_i\rangle \underbrace{\langle v_i|b\rangle}_{\substack{\text{c-number} \\ c_i}}$$

$$= \sum_i c_i \hat{A}|v_i\rangle$$ changed bases

Eigenvalues of Projector operators

$$\hat{P}|u\rangle = \lambda|u\rangle$$

$$\hat{P}^2|u\rangle = \hat{P}(\hat{P}|u\rangle) = \hat{P}\lambda|u\rangle = \lambda\hat{P}|u\rangle$$

$$= \lambda^2|u\rangle$$

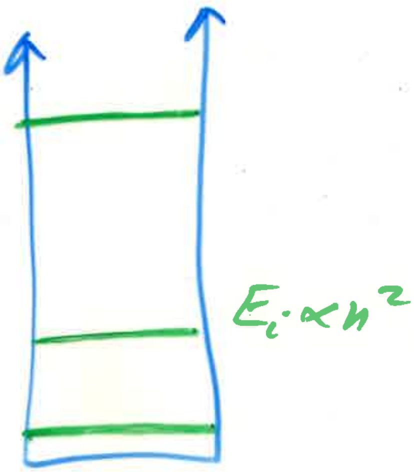
$$= \hat{P}|u\rangle = \lambda|u\rangle$$

$$\Rightarrow (\lambda^2 - \lambda)|u\rangle = 0$$ $|u\rangle \neq |0\rangle$ eigenvectors never zero

$$\Rightarrow \lambda(1-\lambda) = 0 \qquad \lambda = 0 \text{ or } \lambda = 1$$

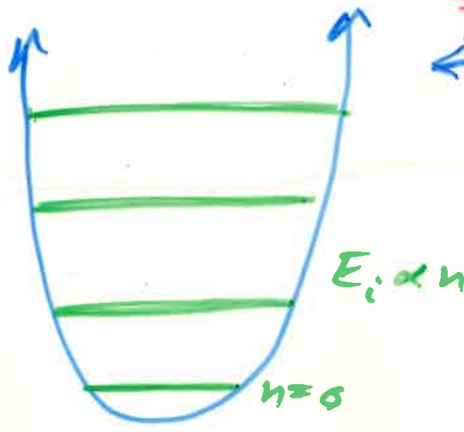
gauge

Infinite square well



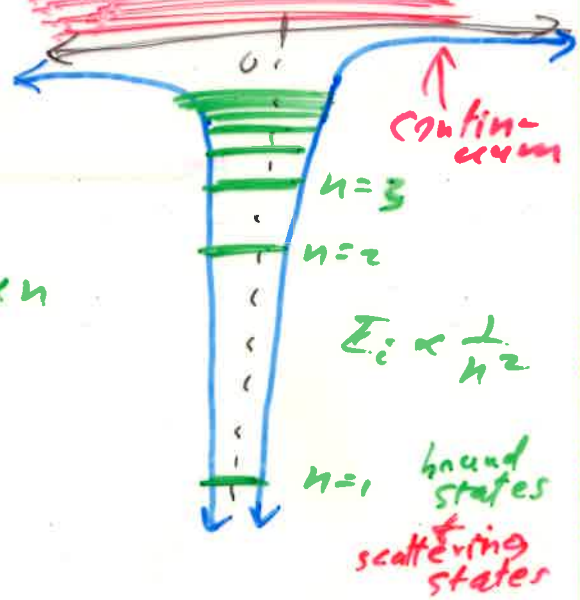
only bound states

Simple Harmonic Oscillator



only bound states

Hydrogen atom



$$V(r) = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}$$

Hydrogen Balmer Series

- 3 → 2 Red
- 4 → 2 Cyan
- 5 → 2 violet
- 6 → 2 dark violet

electrons = points

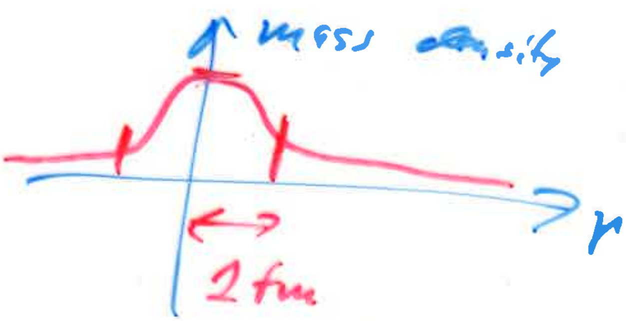
$R_e < 10^{-18}$ m

1 attometer

proton

$R_p \sim 10^{-15}$ m = 1 fm

= 1 fermi



S.E. $i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} + \hat{V}\right)\Psi$

one dimension

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\,\Psi(x,t)$$

$\uparrow$ $\hat{p}_x^2$ $\quad \hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\quad$ ↑ still no time dependence

three dimensions

∇^2 Laplacian $= \vec{\nabla}\cdot\vec{\nabla} =$ div. grad.

$$i\hbar \frac{\partial}{\partial t}\Psi(x,y,z,t) = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\Psi(x,y,z,t)$$

$$+ V(x,y,z)\,\Psi(x,y,z,t)$$

$-\hbar^2\nabla^2 = \hat{\vec{p}}\cdot\hat{\vec{p}}$ $\qquad \hat{\vec{p}} = \frac{\hbar}{i}\vec{\nabla}_r$ gradient

coordinate-free form

$$i\hbar \frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(\vec{r})\,\Psi(\vec{r},t)$$

Now any coordinate system

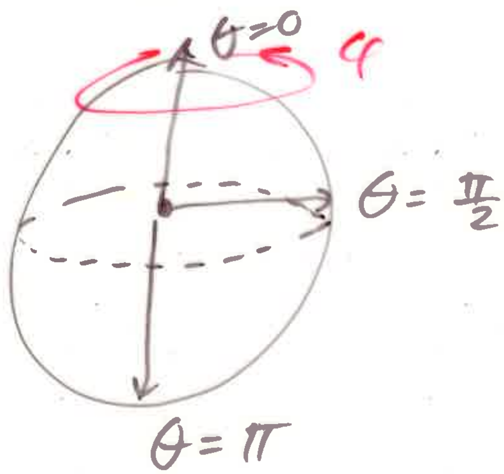
$$\vec{r} = (x,y,z) = (r,\theta,\varphi) = (s,\varphi,z)$$

↑ spherical $\qquad$ ↑ cylindrical

$r = \sqrt{x^2+y^2+z^2}$ $\qquad$ $s = \sqrt{x^2+y^2}$

Differential operator	Acts on	Results in	example
gradient $\vec{\nabla}$	scalar field	vector field	$\vec{\nabla}\Phi(\vec{r}) = -\vec{E}(\vec{r})$ (Φ: electric potential = voltage)
divergence $\vec{\nabla}\cdot$	vector field	scalar field	$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$
curl $\vec{\nabla}\times$ ($\vec{\nabla}$: del)	vector field	different vector field	$\vec{\nabla}\times\vec{A}(\vec{r}) = \vec{B}(\vec{r})$ ($\vec{A}$: vector potential)
Laplacian $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$ (∇^2: 2nd order differential operator)	scalar field	different scalar field	$\nabla^2\Phi(\vec{r}) = \frac{-\rho(\vec{r})}{\epsilon_0}$

Physics definitions of polar and azimuthal angles



$0 \le \theta \le \pi$

$0 \le \varphi \le 2\pi$