Casimir Operators $\quad \hat{L}_z \, , \hat{L}^2$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\left.\begin{aligned} \hat{L}_x^\dagger &= \hat{L}_x \\ \hat{L}_y^\dagger &= \hat{L}_y \\ \hat{L}_z^\dagger &= \hat{L}_z \end{aligned}\right\} \text{Hermit.}$$

$$[\hat{L}_j , \hat{L}_k] = i\hbar\, \epsilon_{jkp}\, \hat{L}_p$$

$\uparrow$ Levi-Civita tensor

$$(\vec{A}\times\vec{B})_k = \vec{c}_k$$

$$\sum_i \sum_j \epsilon_{ijk} A_i B_j$$

$$\epsilon_{123} = +1 = \epsilon_{231} = \epsilon_{312}$$

$$\epsilon_{321} = -1 = \epsilon_{213} = \epsilon_{132}$$

$$\epsilon_{222} = 0 \, , \epsilon_{113} = 0$$

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

$$\hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} \stackrel{?}{=} \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B}$$

$$\hat{A}\hat{B}\hat{C} - \cancel{\hat{A}\hat{C}\hat{B}} + \cancel{\hat{A}\hat{C}\hat{B}} - \hat{C}\hat{A}\hat{B}$$

$$[\hat{L}^2, \hat{L}_x] = [(\cancel{\hat{L}_x^2} + \hat{L}_y^2 + \hat{L}_z^2), \hat{L}_x]$$

$$= [\hat{L}_y^2, L_x] + [\hat{L}_z^2, L_x]$$

$$= \hat{L}_y \underbrace{[\hat{L}_y, L_x]}_{-i\hbar L_z} + \underbrace{[\hat{L}_y, \hat{L}_x]}_{-i\hbar L_z} \hat{L}_y + \hat{L}_z \underbrace{[\hat{L}_z, \hat{L}_x]}_{+i\hbar L_y} + \underbrace{[\hat{L}_z, \hat{L}_x]}_{+i\hbar L_y} \hat{L}_z$$

$$= 0 = [\hat{L}^2, L_y] = [\hat{L}^2, \hat{L}_z]$$

Prepare a state $|\psi\rangle$

① Measure $\hat{L}^2|\psi\rangle = 2\hbar^2|\psi\rangle = \ell(\ell+1)\hbar^2|\psi\rangle$

$\Rightarrow \ell = 1$

② Measure $\hat{L}_x|\psi\rangle = \boxed{-1}\hbar|\psi\rangle = m\hbar|\psi\rangle$

$\Rightarrow m = -1$

③ $\hat{L}_x\hat{L}^2\hat{L}^2\hat{L}_x\hat{L}\hat{L}_x\hat{L}^2|\psi\rangle = ?|\psi\rangle$

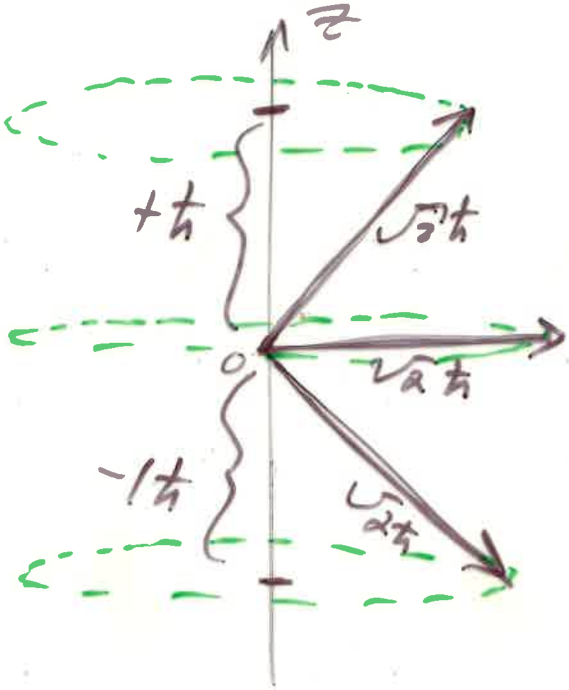
$\hat{L}_y|\psi\rangle = \underline{\ 0\ }\hbar|\varphi\rangle \leftarrow$ new state (ket)

+1, 0, −1 each with 33% chance

$\hat{L}_x|\varphi\rangle = 0\hbar|\varphi\rangle$

$\hat{L}^2|\varphi\rangle = 2\hbar^2|\varphi\rangle$

$$|\psi_{\ell=1, m_x=-1}\rangle = \frac{1}{\sqrt{3}}|\varphi_{\ell=1, m_y=+1}\rangle + \frac{1}{\sqrt{3}}|\varphi_{\ell=1, m_y=0}\rangle + \frac{1}{\sqrt{3}}|\varphi_{\ell=1, m_y=-1}\rangle$$



$\ell = 1$

$|\vec{L}| = \sqrt{\ell(\ell+1)}\,\hbar$

$= \sqrt{2}\hbar \approx 1.414\hbar$

$\hat{L}_z = +1\hbar,\ 0\hbar,\ -1\hbar$

$\ell = 0$ (s state)

$\hat{L}_x \to 0$

$\hat{L}_y \to 0$

$\hat{L}_z \to 0$

$\hat{L}^2 \to 0$

prepare state $\ell = 1$

measure $L_z \to$ get $+1\hbar$

measure $L_y \to$ get $= -1\hbar$

measure L_z again $\to$ get $0\hbar$

measure $\hat{L}^2 \to$ get $2\hbar^2$ always.

$$\vec{L} = \hat{L}_x \vec{e}_x + \hat{L}_y \vec{e}_y + \hat{L}_z \vec{e}_z$$

It is possible to find a state $|\psi\rangle$ which is simultaneously an eigenstate of $\hat{L}^2$, $\hat{L}_z$.

$$\hat{L}^2|\psi\rangle = \ell(\ell+1)\hbar^2|\psi\rangle \quad , \quad \hat{L}_z|\psi\rangle = m\hbar|\psi\rangle$$

$$\hat{L}^2|Y_\ell^m(\theta,\varphi)\rangle = \ell(\ell+1)\hbar^2|Y_\ell^m(\theta,\varphi)\rangle$$

$$\hat{L}|Y_\ell^m(\theta,\varphi)\rangle = m\hbar|Y_\ell^m(\theta,\varphi)\rangle =$$

$$|Y_\ell^m(\theta,\varphi)\rangle = |\ell,m\rangle$$

$m, \ell \in$ Integer $\qquad \ell = 0,1,2,3,\ldots$

$$m = \underbrace{-\ell, -\ell+1, \ldots, 0, \ell-1, \ell}_{2\ell+1}.$$

Ladder Operator

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \quad , \quad \hat{L}_- = \hat{L}_x - i\hat{L}_y$$

$$\hat{L}_+^\dagger = \hat{L}_x^\dagger - i\hat{L}_y^\dagger = \hat{L}_x - i\hat{L}_y = \hat{L}_- \quad // \quad \hat{L}_-^\dagger = \hat{L}_+$$

Commutator of $\hat{L}_z$ and $\hat{L}_\pm$.

$$[\hat{L}_z, \hat{L}_\pm] = [\hat{L}_z, (\hat{L}_x \pm i\hat{L}_y)]$$

$$= \underbrace{[\hat{L}_z, \hat{L}_x]}_{i\hbar\hat{L}_y} \pm i\underbrace{[\hat{L}_z, \hat{L}_y]}_{-i\hbar\hat{L}_x}$$

$$= \hbar[\pm\hat{L}_x + i\hat{L}_y] = \pm\hbar[\hat{L}_x \pm i\hat{L}_y] = \pm\hbar L_\pm$$

$$[\hat{L}^2, \hat{L}_\pm] = 0$$

$|\psi\rangle$ is a state, $\hat{L}_+|\psi\rangle = |\varphi\rangle$ ← new state

Claim: If $|\psi\rangle$ is a simultaneous eigenstate of both $\hat{L}^2$ and $\hat{L}_z$ operators, then $|\varphi\rangle$ is also a simultaneous eigenstate of $\hat{L}^2$ and $\hat{L}_z$.

If $\hat{L}^2|\psi\rangle = \ell(\ell+1)\hbar^2|\psi\rangle$, $\hat{L}_z|\psi\rangle = m\hbar|\psi\rangle$

then $\hat{L}^2|\varphi\rangle = \ell(\ell+1)\hbar^2|\varphi\rangle$, $\hat{L}_z|\varphi\rangle = (m+1)\hbar|\varphi\rangle$ or $0|\varphi\rangle$

$$\hat{L}^2|\varphi\rangle = \hat{L}^2\left(\hat{L}_+|\psi\rangle\right) = \hat{L}_+\hat{L}^2|\psi\rangle$$

$$= \hat{L}_+\left[\ell(\ell+1)\hbar^2|\psi\rangle\right] = \ell(\ell+1)\hbar^2\hat{L}_+|\psi\rangle$$

$$= \ell(\ell+1)\hbar^2|\varphi\rangle$$