$$\hat{L}^2|\varphi\rangle = \hat{L}^2(\hat{L}_+|\psi\rangle) = \hat{L}_+\hat{L}^2|\psi\rangle$$

$$= \hat{L}_+\left[\ell(\ell+1)\hbar^2|\psi\rangle\right] = \ell(\ell+1)\hbar^2\hat{L}_+|\psi\rangle$$

$$= \ell(\ell+1)\hbar^2|\varphi\rangle$$

$$\hat{L}_z|\varphi\rangle = \hat{L}_z(\hat{L}_+|\psi\rangle) = \underline{(\hat{L}_z\hat{L}_+)}|\psi\rangle$$

$$= (\hbar\hat{L}_+ + \hat{L}_+\hat{L}_z)|\psi\rangle$$

$$= \hbar\hat{L}_+|\psi\rangle + \hat{L}_+ m\hbar|\psi\rangle = \hbar(1+m)\hat{L}_+|\psi\rangle$$

$$= \hbar(1+m)|\varphi\rangle \quad \overset{?}{=} \quad 0|\varphi\rangle \quad \text{if } m = m_{max}$$

The raising process must eventually end.

$$\hat{L}_+|\ell, m_{max}\rangle = 0 \quad \text{since} \quad \boxed{m^2 \le \ell(\ell+1)}$$

Proof: $\langle\hat{L}^2\rangle = \langle\psi|\hat{L}^2|\psi\rangle = \langle\psi|\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2|\psi\rangle$

$$= \underbrace{\langle\psi|\hat{L}_x^2|\psi\rangle} + \langle\psi|\hat{L}_y^2|\psi\rangle + \langle\psi|\hat{L}_z^2|\psi\rangle$$

$$= (\langle\psi|\hat{L}_x)(\hat{L}_x|\psi\rangle) + \quad \vdots$$

$$= \langle x|x\rangle \qquad \langle\alpha|\alpha\rangle + m^2\hbar^2\cancel{\langle\psi|\psi\rangle}^{1}$$

$$\ell(\ell+1)\hbar^2 \cancel{\langle\psi|\psi\rangle}^{1} = m^2\hbar^2 + \underbrace{b^2}_{\geq 0} + \underbrace{a^2}_{\geq 0}$$

$$\hat{L}_z|\ell, m_{max}\rangle = ? = q\hbar|\ell, m_{max}\rangle$$

$$\hat{L}_+\hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y)$$

$$= \underbrace{\hat{L}_x^2 + \hat{L}_y^2}_{\hat{L}^2 - \hat{L}_z^2} - i\underbrace{(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)}_{[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z}$$

$$\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 - i(i\hbar)\hat{L}_z$$

$$\hat{L}^2 = \hat{L}_{\underset{-}{+}}\hat{L}_{\underset{+}{-}} + \hat{L}_z^2 \underset{+}{-} \hbar\hat{L}_z$$

$$\hat{L}^2|\ell, m_{max}\rangle = (\hat{L}_{\underset{-}{+}}\hat{L}_{\underset{+}{-}} + \hat{L}_z^2 \underset{+}{-} \hbar\hat{L}_z)|\ell, m_{max}\rangle$$

$$\ell(\ell+1)\hbar^2|\ell, m_{max}\rangle = (0 + q^2\hbar^2 + \hbar q\hbar)|\ell, m_{max}\rangle$$

$$= q(q+1)\hbar^2|\ell, m_{max}\rangle$$

$q = \ell$

$m_{max} = \ell$ $\qquad$ $m_{min} = -\ell$

$$m \in \underbrace{\{-\ell, -\ell+1 \dots -1, 0, +1, \dots \ell-1, \ell\}}_{2\ell+1}$$

There are $N^{\geq 0}$ integer steps between $-\ell$ and $+\ell$.

$$-\ell + N = +\ell \Rightarrow N = 2\ell \Rightarrow \ell = \frac{N}{2}$$

$$\ell \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \dots\}$$

orbital angular momentum	spin angular momentum
ℓ must an integer	ℓ can be any in the set.

Cohen-Tannoudji

It is possible to find simultaneous eigenvectors

$$[\hat{L}_z, \hat{L}^2] = 0, \quad [\hat{H}, \hat{L}_z] = 0, \quad [\hat{H}, \hat{L}^2] = 0$$

$$\hat{H}|\psi_{n\ell m}\rangle = E_{n\ell}|\psi_{n\ell m}\rangle$$

$$\hat{L}^2|\psi_{n\ell m}\rangle = \ell(\ell+1)\hbar^2|\psi_{n\ell m}\rangle$$

$$\hat{L}_z|\psi_{n\ell m}\rangle = m\hbar|\psi_{n\ell m}\rangle$$