# Symmetry + Conservation Laws

① Cyclic variables

If $L$ does not depend on some generalized coordinate $q_j$, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \cancel{\frac{\partial L}{\partial q_j}}^{\to 0} = 0 \implies \frac{\partial L}{\partial \dot{q}_j} = \text{constant}$$

$P_j \equiv \frac{\partial L}{\partial \dot{q}_j}$ is called the canonical "momentum" conjugate $q_j$

If $q_j$ is not a length, then $p_j$ is not a momentum.

e.g. If $L$ does not depend on the azimuthal angle $\varphi$, but $L$ does contain a kinetic term $\frac{m}{2} r^2 \sin^2\theta\, \dot{\varphi}^2$, then

$$P_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = m(r\sin\theta)^2\dot{\varphi} = \text{angular momentum}$$

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② If the transformation $q_j \to q_j + \delta q_j$ ; $\dot{q}_j \to \dot{q}_j + \frac{d}{dt}\delta q_j$ leaves the action invariant, then there is a constant of the motion

$$\text{Action } I = \int_{t_1}^{t_2} L(q,\dot{q},t)\, dt$$

Action is stationary.

$\underline{\delta I = 0}$

$$\delta I = \int_{t_1}^{t_2} \left( \sum_j \frac{\partial L}{\partial q_j} \delta q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt = 0$$

use $\frac{\partial L}{\partial q_j} = + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j}$

$$\delta I = \int_{t_1}^{t_2} \sum_j \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \delta q_j \Big) dt = 0$$

$$= \int_{t_1}^{t_2} \frac{d}{dt} \left( \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) dt = \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2} = 0$$

$$\Longrightarrow \quad \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j = \text{constant}$$

Two particles in one dimension

e.g. $L = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 + V(x_1 - x_2)$

no cyclic variables

Transformation: $x_1 \to x_1 + a \Rightarrow \delta x_1 = a$

$x_2 \to x_2 + a \Rightarrow \delta x_2 = a$

$\dot{x}_1 \to \dot{x}_1 \Rightarrow \delta \dot{x}_1 = 0$

$$\sum_{j=1}^{2} \frac{\partial L}{\partial \dot{x}_j} \delta x_j = (m_1 \dot{x}_1 + m_2 \dot{x}_2) a = \text{constant}$$
$$= (p_1 + p_2) a$$

Close connection between translational invariance and linear momentum conservation.

$\Longrightarrow$ Noether's Theorem (see ch 13)

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③ Changes in the action $I$ are the total time derivative of something

$$L(q_j+\delta q_j, \dot{q}_j+\delta\dot{q}_j, t) - L(q_j, \dot{q}_j, t) = \frac{d\Lambda}{dt}$$

$$\Longrightarrow \delta I \neq 0$$

$$\delta I = \int_{t_1}^{t_2} \frac{d\Lambda}{dt}\,dt \Longrightarrow \sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j - \Lambda = \text{constant}$$

e.g. $L = \frac{m}{2} v^2 + q(\Phi - \vec{v}\cdot\vec{A})$

$\Phi(\vec{r}, t)$ - scalar potential = voltage

$\vec{A}(\vec{r}, t)$ - vector potential

$\Lambda(\vec{r}, t)$ -

$$\vec{E}(\vec{r}, t) = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B}(\vec{r}, t) = \vec{\nabla}\times\vec{A}$$

Transformation - Gauge transformation

$$\Phi \to \Phi + \frac{\partial\Lambda}{\partial t} \qquad \vec{A} \to \vec{A} - \vec{\nabla}\Lambda$$

$$\delta L = q\left(\frac{\partial\Lambda}{\partial t} + \vec{v}\cdot\vec{\nabla}\Lambda\right) = q\frac{d\Lambda}{dt}$$

# Hamilton's Principle

A physical system moves such that the action $I = \int_{t_1}^{t_2} L\, dt$ is stationary, $\delta I = 0$, for the actual path.

Suppose $q_j(t,0)$ are the solutions we seek that extremize $I$.

$$q_j(t,\alpha) \equiv q_j(t,0) + \alpha\, \eta_j(t)$$

$$\dot{q}_j(t,\alpha) \equiv \dot{q}_j(t,0) + \alpha\, \dot{\eta}_j(t)$$

$\eta_i$ are independent, $\eta_j(t_1) = 0 = \eta_j(t_2) \leftarrow$

$\eta_j$ vanishes at the end points.

$$\delta I = \frac{\partial I}{\partial \alpha}\delta\alpha = \int_{t_1}^{t_2} \sum_j \Big( \frac{\partial L}{\partial q_j} \underbrace{\frac{dq_j}{d\alpha}}_{\uparrow\, \eta_j} \delta\alpha + \frac{\partial L}{\partial \dot{q}_j} \underbrace{\frac{d\dot{q}_j}{d\alpha}}_{\uparrow\, \dot{\eta}_j} \delta\alpha \Big) dt$$

$$= \int_{t_1}^{t_2} \sum_j \Big( \frac{\partial L}{\partial q_j}\eta_j + \underline{\frac{\partial L}{\partial \dot{q}_j}\dot{\eta}_j} \Big)\delta\alpha\, dt$$

$\uparrow$ integrate by parts

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j}\frac{d\eta_j}{dt}\,dt = \frac{\partial L}{\partial \dot{q}_j}\eta_j\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\eta_j\,dt$$

zero since $\eta_j(t_1) = 0 = \eta_j(t_2)$

$$\delta I = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\eta_j\,\delta\alpha\,dt = 0$$

$$\Longrightarrow \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad \forall j$$

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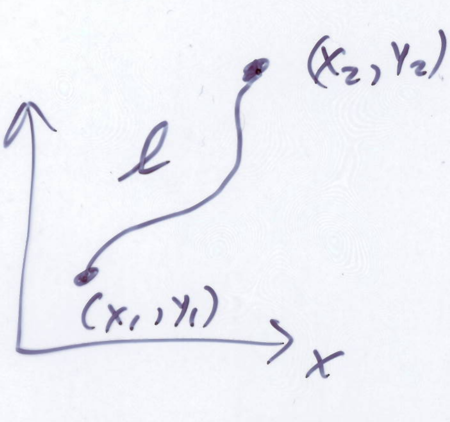
Examples from the calculus of variations

① Shortest distance between two points in the plane ($xy$-plane)

arc length: $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$\equiv \sqrt{1 + \dot{y}^2}\,dx$$




$$l = \int_1^2 ds = \int_{x=x_1}^{x_2} \sqrt{1 + \dot{y}^2}\,dx$$

Integrand $f = \sqrt{1 + \dot{y}^2}$, $f[y(x), \dot{y}(x), \cancel{x}] =$

Extremize (minimize) Length

$$\delta l = 0 \Longrightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$ Euler Lagrange Equation

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\delta l = 0 \Longrightarrow \frac{d}{dx}\left[\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right] = 0 \Longrightarrow \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = \underset{\text{constant}}{c}$$

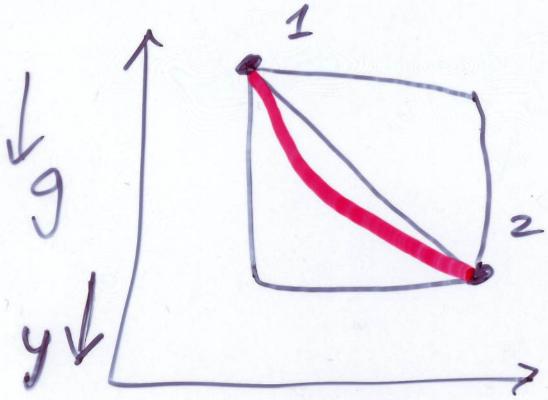
solve for $\dot{y}$

$$\dot{y} = \frac{c}{\sqrt{1-c^2}} \equiv a$$ another constant

$$\dot{y} = \frac{dy}{dx} = a \Longrightarrow y = ax + b$$ straight line

pick $\underline{a}$ and $\underline{b}$ to go through the endpoints.

# Brachistochrone




Minimize travel time from 1 to 2

$$t = \int_1^2 \frac{ds}{v} \qquad \delta t = 0$$

Get $v$ from energy conservation $\frac{1}{2}mv^2 = mgy$

$$v = \sqrt{2gy} \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1+\dot{y}^2}\, dx$$

$$t = \int_{x_1}^{x_2} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\, dx = \int_{x_1}^{t_2} f[y, \dot{y}, x]\, dx$$

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$$f = \sqrt{\frac{1+\dot{y}^2}{2gy}} \qquad \frac{\partial f}{\partial y} = \sqrt{\frac{1+\dot{y}^2}{2g}} \left( \frac{-1}{2y^{3/2}} \right)$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{1}{\sqrt{2gy}} \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$

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$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$$

$$\sqrt{\frac{1+\dot{y}^2}{2g}} \left( \frac{-1}{2y^{3/2}} \right) - \frac{d}{dx}\left[ \frac{1}{\sqrt{2gy}} \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} \right] = 0$$

Two ways to do this more simply.

Beltrami Identity - 2nd form of Euler-Lagrange equation

$$* \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0} \qquad \Bigg| \qquad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

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Notice

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial \dot{y}}\frac{d\dot{y}}{dx}$$

$$= \frac{\partial f}{\partial x} + \underbrace{\frac{\partial f}{\partial y}\dot{y}}_{\text{solve for this}} + \frac{\partial f}{\partial \dot{y}}\ddot{y}$$

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$$\frac{\partial f}{\partial y}\dot{y} = \frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial \dot{y}}\ddot{y}$$

$$* \quad \frac{\partial f}{\partial y}\dot{y} = \left(\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\dot{y}$$

subtract

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$$0 = \frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial \dot{y}}\ddot{y} - \left(\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\dot{y}$$

$$\boxed{0 = -\frac{\partial f}{\partial x} + \frac{d}{dx}\left[f - \frac{\partial f}{\partial \dot{y}}\dot{y}\right]}$$

second form of Euler-Lagrange

If $\frac{\partial f}{\partial x} = 0$, then

$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = \text{constant}$$