Euler-Lagrange for radial coordinate r

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0$$

$$m\ddot r - m r\dot\theta^2 + \underbrace{\frac{\partial V}{\partial r}}_{-f(r)} = 0$$

$\nearrow$ reduced mass μ

substitute: $\dot\theta = \frac{\ell}{mr^2}$

$$m\ddot r = \frac{\ell^2}{mr^3} + f(r)$$

one-dimensional problem

$$\underset{\displaystyle ma}{\underset{\downarrow}{m\ddot r}} = \underset{\text{centrifugal force}}{\underset{\downarrow}{\frac{\ell^2}{mr^3}}} + \underset{\text{real force}}{\underset{\nwarrow}{f(r)}}$$

$$m\ddot r = -\frac{d}{dr}\Big[\underbrace{\frac{1}{2}\frac{\ell^2}{mr^2} + V}_{\text{effective potential } V_{eff}}\Big]$$

Multiply both sides by integrating factor $\dot r$

$$m\ddot r\dot r = -\frac{d}{dr}\Big[\frac{1}{2}\frac{\ell^2}{mr^2} + V\Big]\dot r = -\frac{d}{dr}\Big[\frac{\ell^2}{2mr^2} + V\Big]\frac{dr}{dt}$$

$$= \frac{d}{dt}\left(\frac{1}{2}m\dot r^2\right)$$

$$\Rightarrow \frac{d}{dt}\Big[\frac{1}{2}m\dot r^2 + \frac{\ell^2}{2mr^2} + V(r)\Big] = 0$$

$$\frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) = \text{constant} = \underset{\substack{\nearrow \\ \text{energy}}}{E}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) = T + V$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} \qquad \text{integrate}$$

$$t = \int_{r'=r_0}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}\left[E - V(r') - \frac{\ell^2}{2mr'^2}\right]}}$$

for a given $V(r)$, this gives $t(r)$. Invert to get $r(t)$.

How do we get $\theta(t)$

$$\ell = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{d\theta}{dt} = \frac{\ell}{mr^2}$$

$$d\theta = \frac{\ell\, dt}{m r^2(t)} \qquad \text{integrate}$$

$$\theta(t) = \int_{t'=0}^{t} \frac{\ell\, dt'}{m[r(t')]^2} + \theta_0$$

We have two second-order differential equations $\Longrightarrow$ need 4 initial conditions to fix $r(t)$ and $\theta(t)$.

Could have chosen: $r_0, \theta_0, \dot{r}_0$ & $\dot{\theta}_0$.

Instead we have: r_0, θ_0, E, ℓ.

Particle speed

$$E = T + V = \frac{1}{2} m v^2 + V(r)$$

$$v = \sqrt{\frac{2}{m}\left[E - V(r)\right]}$$

$$\dot{r} = v_r = \sqrt{\frac{2}{m}\Big[\underbrace{E - V(r) - \frac{\ell^2}{2mr^2}}_{E - V_{eff}}\Big]}$$

$$v_\theta = r\dot{\theta} = \sqrt{v^2 - v_r^2} = \sqrt{\frac{2\ell^2}{m \cdot 2mr^2}} = \frac{\ell}{mr}$$

Inverse square force law $\vec{f} = \frac{-k}{r^2}\hat{r}$

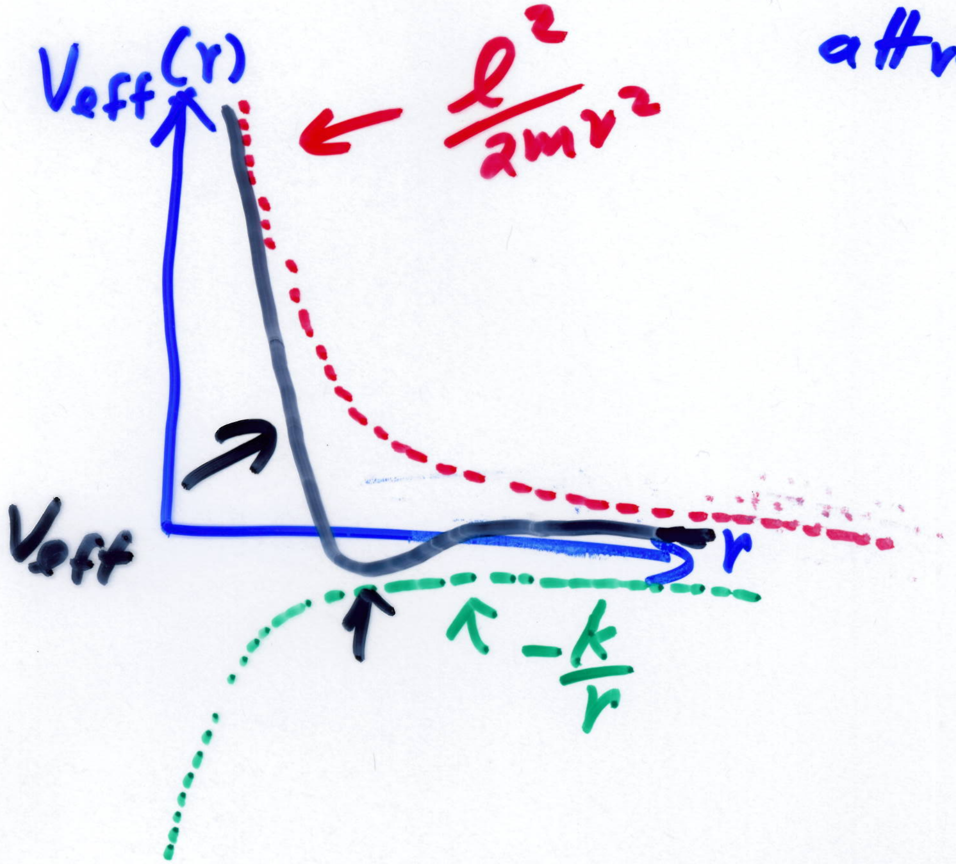
$k > 0$ is attractive, like gravity.

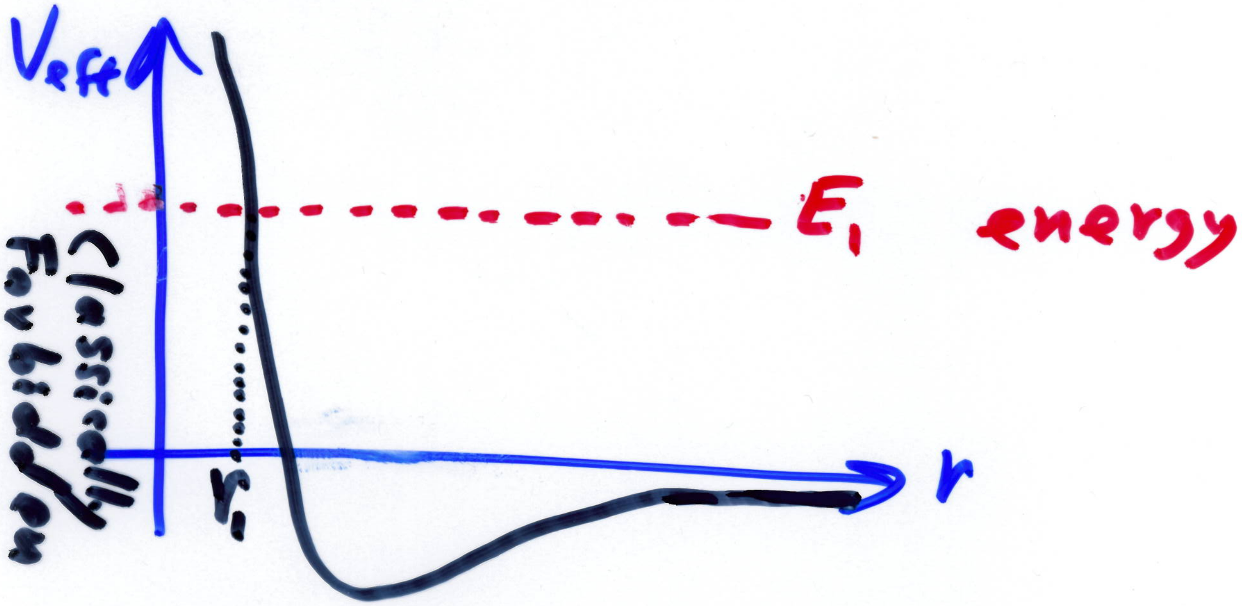
$$V(r) = -\frac{k}{r}$$

Effective potential $V_{eff} = -\frac{k}{r} + \frac{\ell^2}{2mr^2}$

attractive (arrow to $-\frac{k}{r}$)

repulsive (arrow to $\frac{\ell^2}{2mr^2}$)



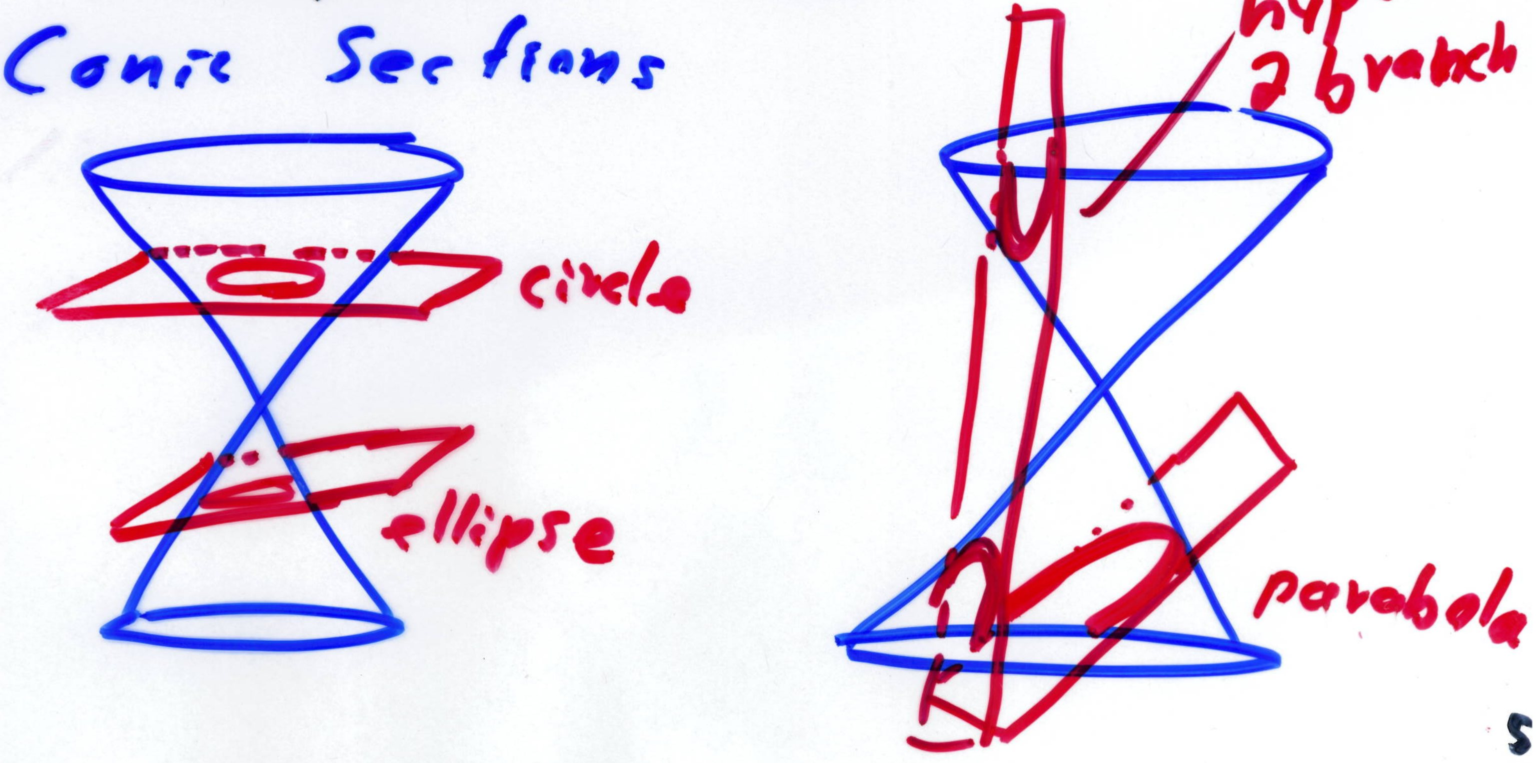


If $E < V_{eff}$ then $\dot{r} = v_r$ is imaginary. r_1 is a turning point - distance of closest approach. For this E_1, there is no upper limit to r

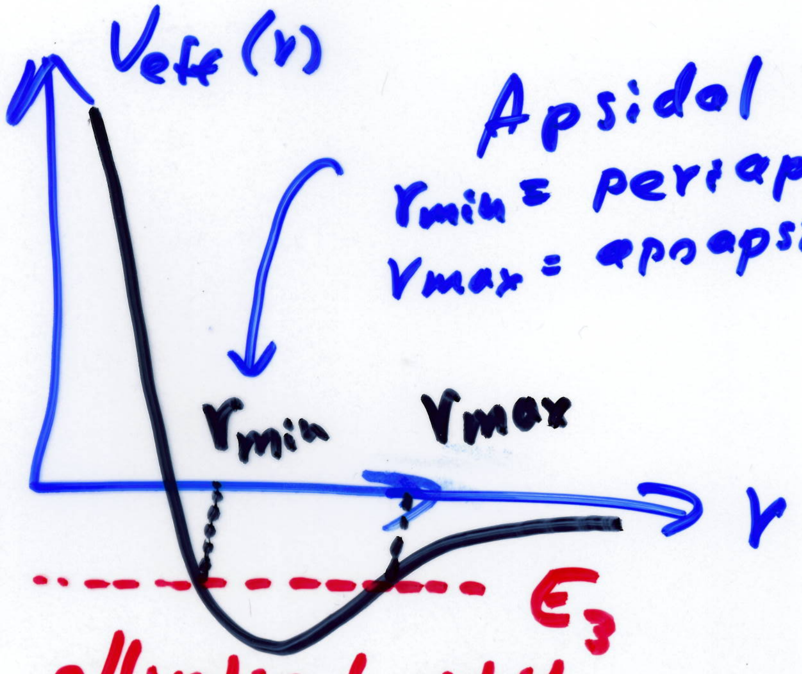
$\Rightarrow$ motion is unbound.

$\Rightarrow$ hyperbolic orbit.

Conic Sections



b is the lever arm for angular momentum. $\ell = m\vec{v}_0 b$



Apsidal distances

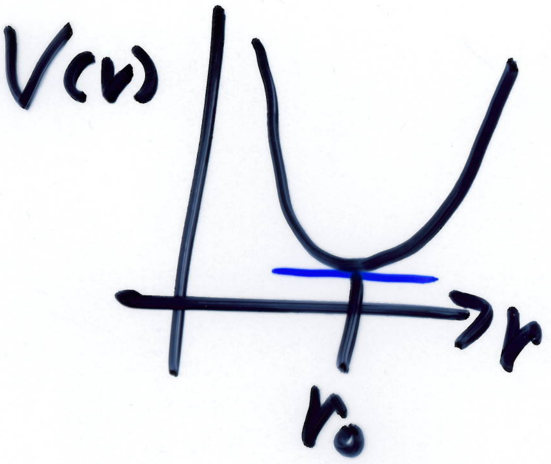
r_{min} = periapsis, perihelion, perigee

r_{max} = apoapsis, aphelion, apogee

$E_2 = 0$ parabolic orbit

elliptical orbit (bound)

Any stable potential looks parabolic close to the minimum r value.



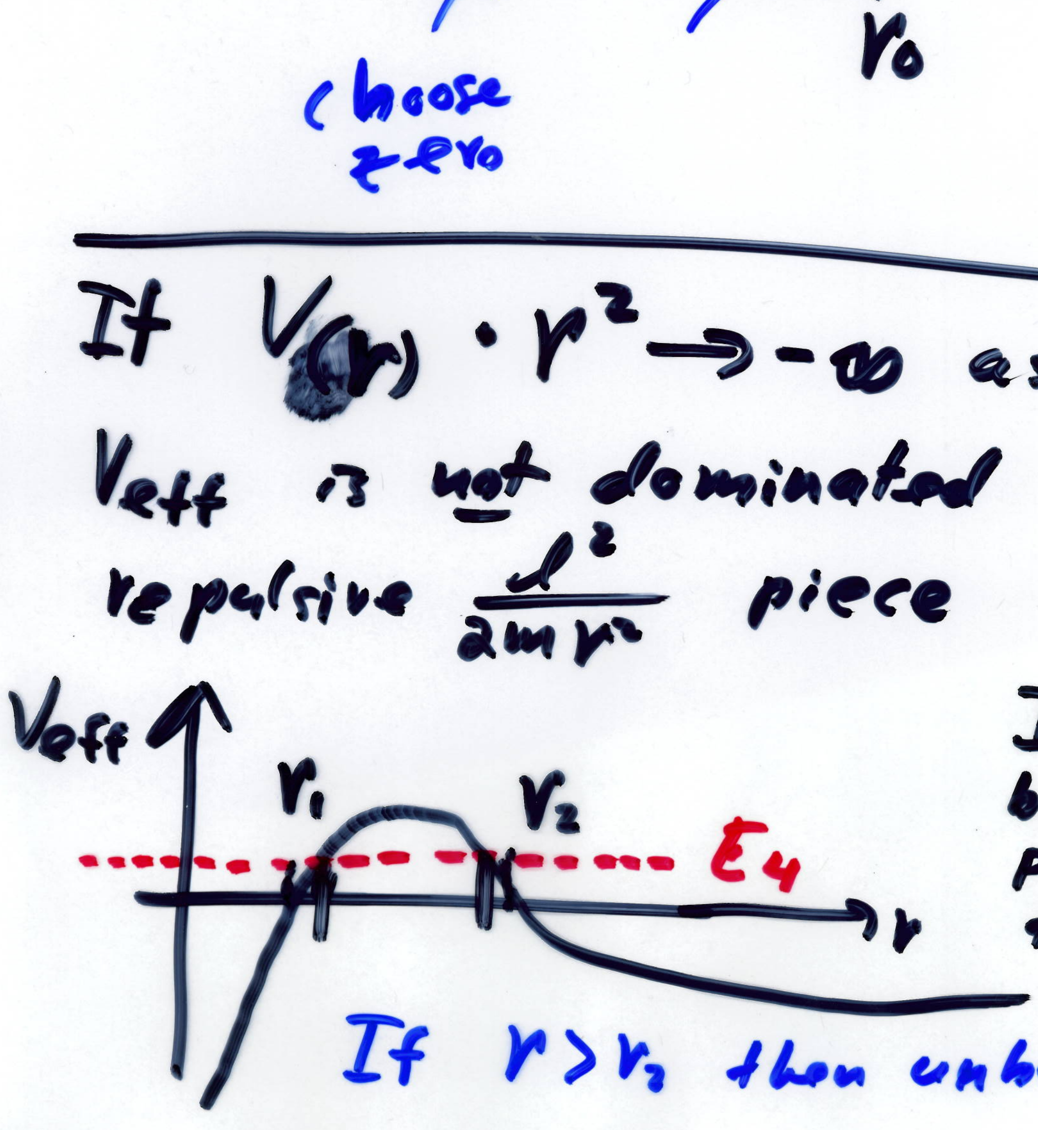
Taylor Expansion

$$V(r) = \underbrace{V(r_0)}_{\text{choose zero}} + \cancel{\left.\frac{dV}{dr}\right|_{r_0}}^{\,0} \cdot (r-r_0) + \frac{1}{2!}\left.\frac{d^2V}{dr^2}\right|_{r_0}(r-r_0)^2$$

$$+ \cdots$$

If $V(r) \cdot r^2 \to -\infty$ as $r \to 0$

V_{eff} is <u>not</u> dominated by the repulsive $\frac{l^2}{2mr^2}$ piece as $r \to 0$



If $r < r_1$ then bound motion — particle will pass through the force center.

If $r > r_2$ then unbound motion