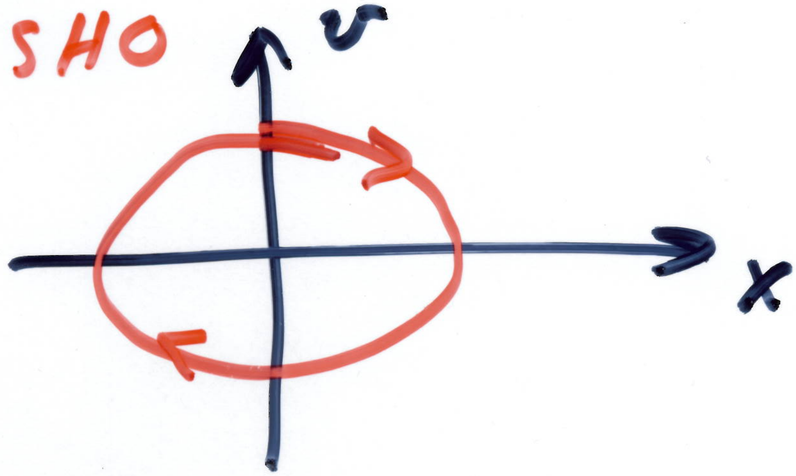
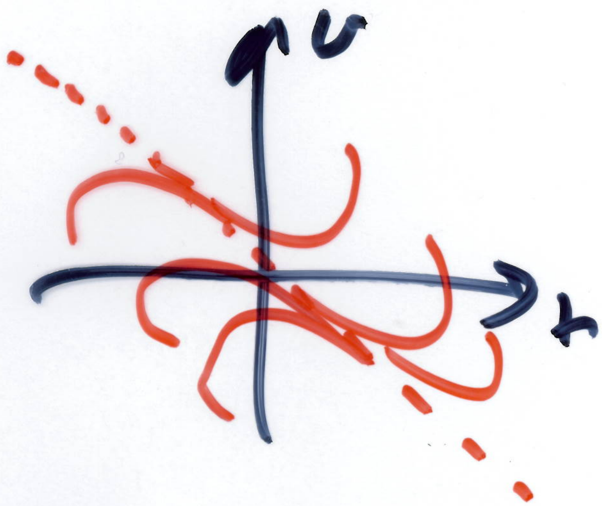
Phase Space

SHO

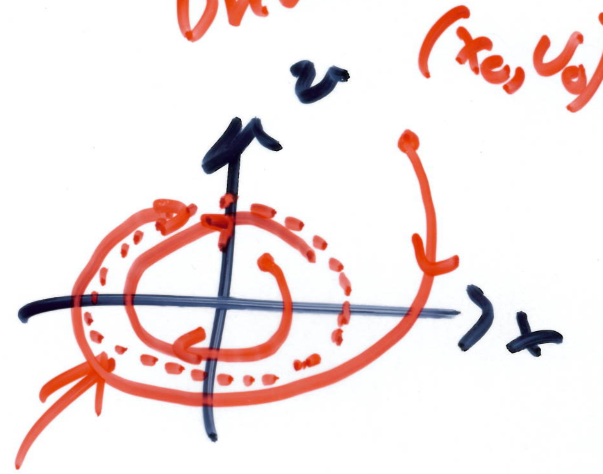


Damped

overdamped



Damped Driven



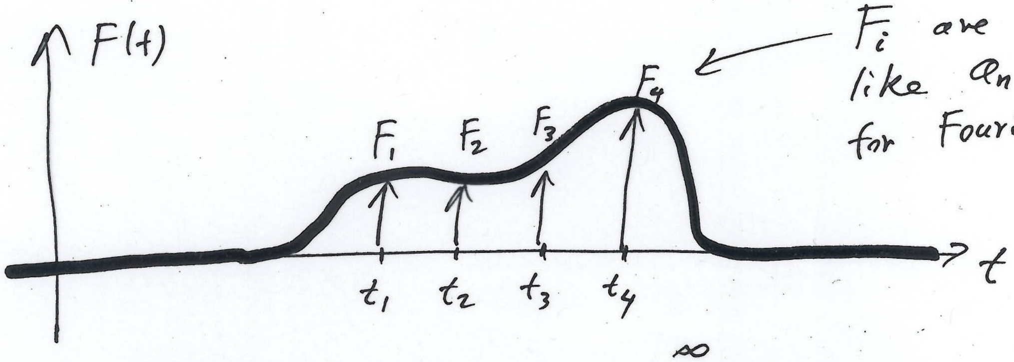
$x_p(t)$

limit cycle

$x_c(t) = e^{-\beta t}\{ \cdots \}$

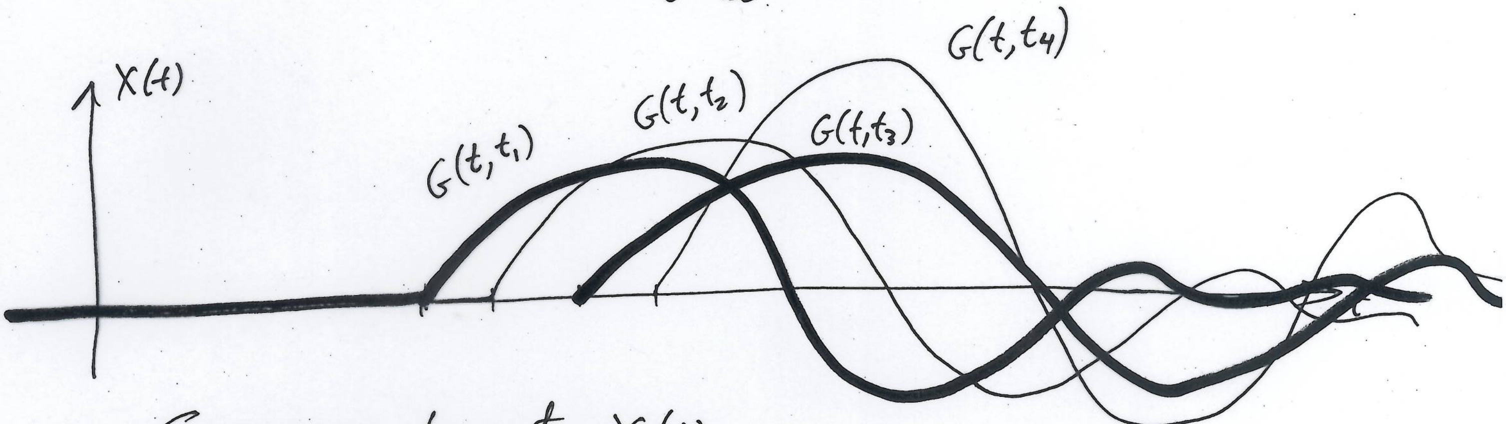
An arbitrary function $F(t)$ can be decomposed as:

1) sines + cosines (Fourier)
2) delta functions (Green)



F_i are amplitudes, like a_n and b_n for Fourier series

$$F(t) = \sum_{i=1}^{4} F_i\,\delta(t-t_i) \longrightarrow \int_{t'=-\infty}^{\infty} F(t')\,\delta(t-t')\,dt'$$



Superpose to get $X(t)$

$$X(t) = \sum_{i=1}^{4} F_i\,G(t,t_i) \longrightarrow \int_{t'=\infty}^{\infty} F(t')\,G(t,t')\,dt'$$

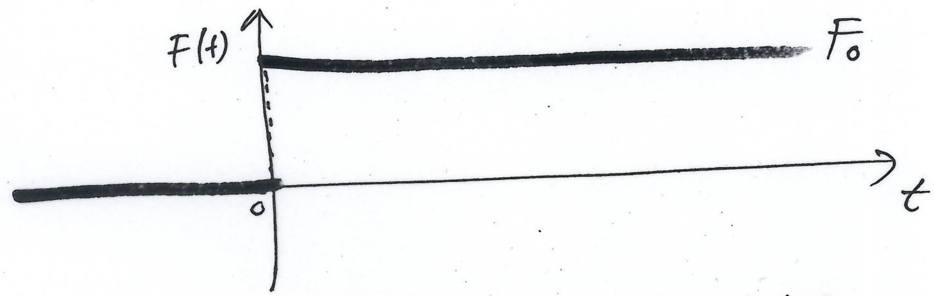
"like" matrix multiplication

This is a convolution integral.

Great! Sort of... If only you knew $G(t,t')$ then this would give you $x(t)$ for any $F(t)$.

Derivation of $G(t,t')$

Consider this forcing function:



$$F(t) = F_0\,\theta(t)$$

We know the complementary solution for $t > 0$.

$$x_c(t) = e^{-\beta t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right]$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ underdamped e.g.

Guess a constant particular solution

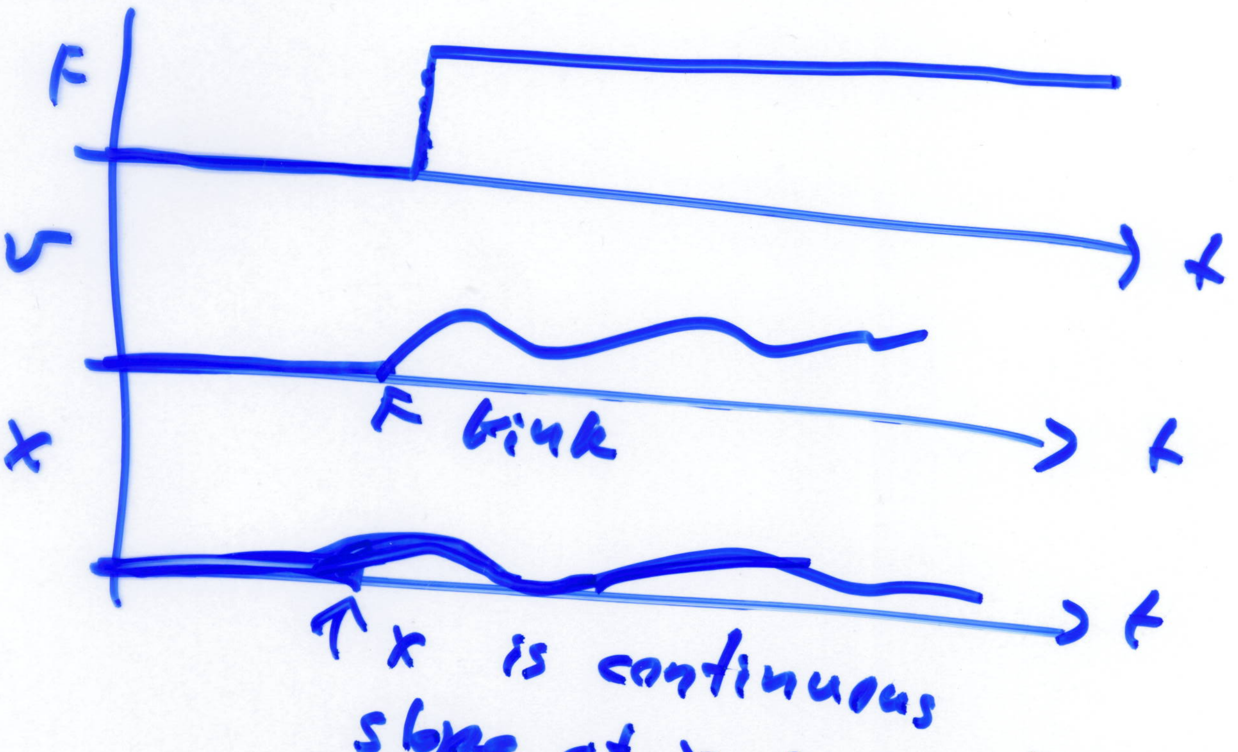
$$x_p(t) = c \qquad \dot{x}_p(t) = 0 \qquad \ddot{x}_p(t) = 0$$

$$\ddot{x}_p(t) + 2\beta\dot{x}_p(t) + \omega_0^2 x_p(t) = \frac{F(t)}{m} = \frac{F_0\,\theta(t)}{m} = \begin{cases} 0, & t<0 \\ \frac{F_0}{m}, & t>0 \end{cases}$$

$$0 + 0 + \omega_0^2 c = \begin{cases} 0, & t<0 \\ \frac{F_0}{m}, & t>0 \end{cases}$$

$$x_p(t) = \begin{cases} 0, & t<0 \\ \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}, & t>0 \end{cases}$$


F
v
x
t
t
t
discontinuity
kink - change in slope


F
v
x
t
t
t
kink
x is continuous
slope of x is continuous

The general solution is $x(t) = x_c(t) + x_p(t)$

$$x(t) = \begin{cases} 0 , & t \leq 0 \\ e^{-\beta t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right] + \frac{F_0}{m\omega_0^2} , & t \geq 0 \end{cases}$$

determine A and B from initial conditions:

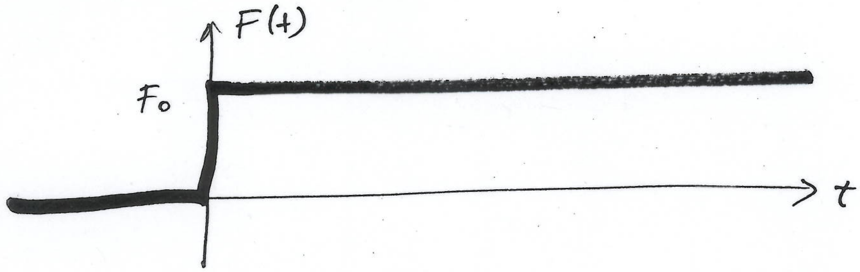
$$\left.\begin{array}{l} x(0) = 0 \\ v(0) = 0 \end{array}\right\} \text{ for example.}$$

$$0 = x(0) = A + \frac{F_0}{m\omega_0^2} \Longrightarrow A = -\frac{F_0}{m\omega_0^2}$$

$$v(t) = \dot{x}(t) = \begin{cases} 0 , & t \leq 0 \\ e^{-\beta t}\left\{-\beta\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right] + \omega_1\left[-A\sin(\omega_1 t) + B\cos(\omega_1 t)\right]\right\} , & t \geq 0 \end{cases}$$

$$0 = v(0) = -\beta A + \omega_1 B \Longrightarrow B = \frac{\beta A}{\omega_1} = -\frac{\beta F_0}{m\omega_1\omega_0^2}$$

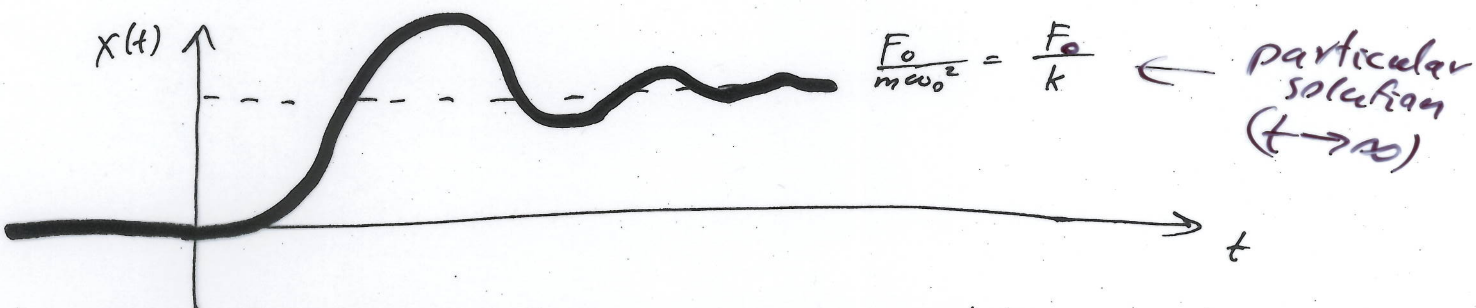
So for this force



$$F(t) = F_0\,\theta(t)$$

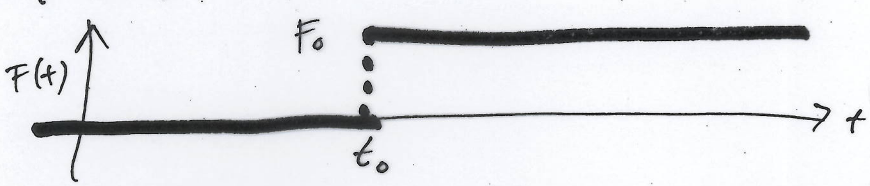
the general solution is

$$X(t) = \begin{cases} 0, & t \le 0 \\ \dfrac{F_0}{m\omega_0^2}\left[1 - e^{-\beta t}\cos(\omega_1 t) - \dfrac{\beta}{\omega_1} e^{-\beta t}\sin(\omega_1 t)\right], & t \ge 0 \end{cases}$$



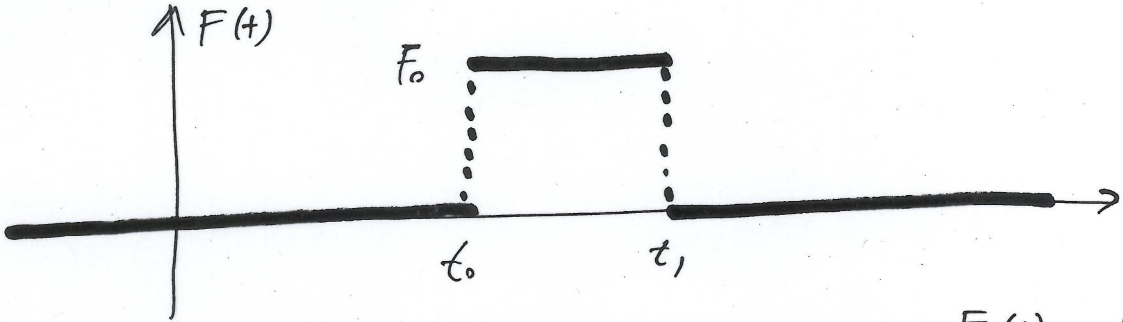
The dashed line is the particular solution $X_p(t)$ which is the steady state (long time) solution. The complementary solution is transient and the damping envelope $e^{-\beta t}$ will effectively wipe them out in 5 e-folding times.

Now, what if the constant force F_0 turns on at t_0 instead of $t = 0$?

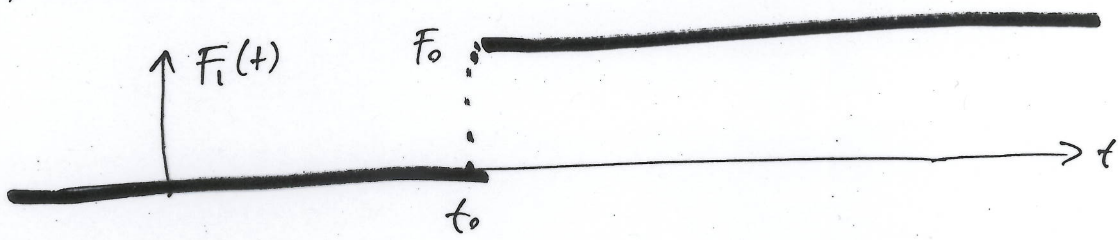


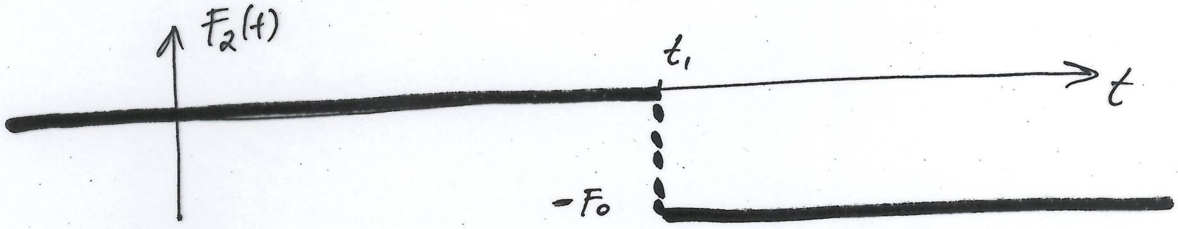
Then in the general solution above, just replace t (really $t-0$) with $(t - t_0)$. So the answer is $X(t - t_0)$.

How about this force?



This is the superposition of two forces $F_1(t)$ and $F_2(t)$.





But we know the general solution for each of these

$$X_1(t) = X(t-t_0) \qquad X_2 = -X(t-t_1)$$

So we superpose the two solutions (isn't linearity great?)

The general solution with square pulse forcing is

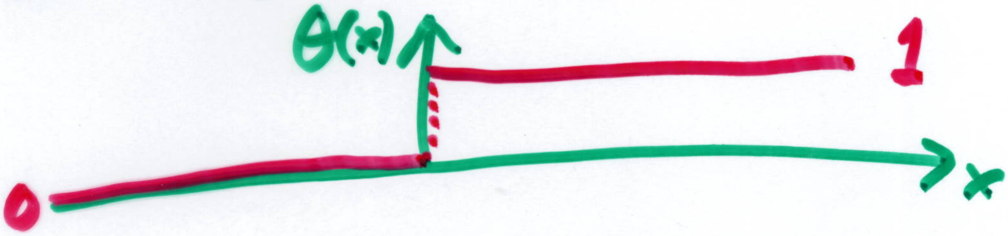
$$X(t-t_0) - X(t-t_1)$$

Now let $(t_1 - t_0) \to 0$ while $F_0 \to \infty$ such that the area under the square pulse is constant, $F_0(t_1 - t_0) = \text{const.}$ The force will become a delta function. Under the same limits, the general solution will become the Green function.

Green Function

$$G(t,t') = \begin{cases} 0 \ , & t \le t' \\ \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1 (t-t')] \ , & t \ge t' \end{cases}$$

or using the Heaviside function $\theta(x)$



$$G(t-t') = \theta(t-t') \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1 (t-t')]$$

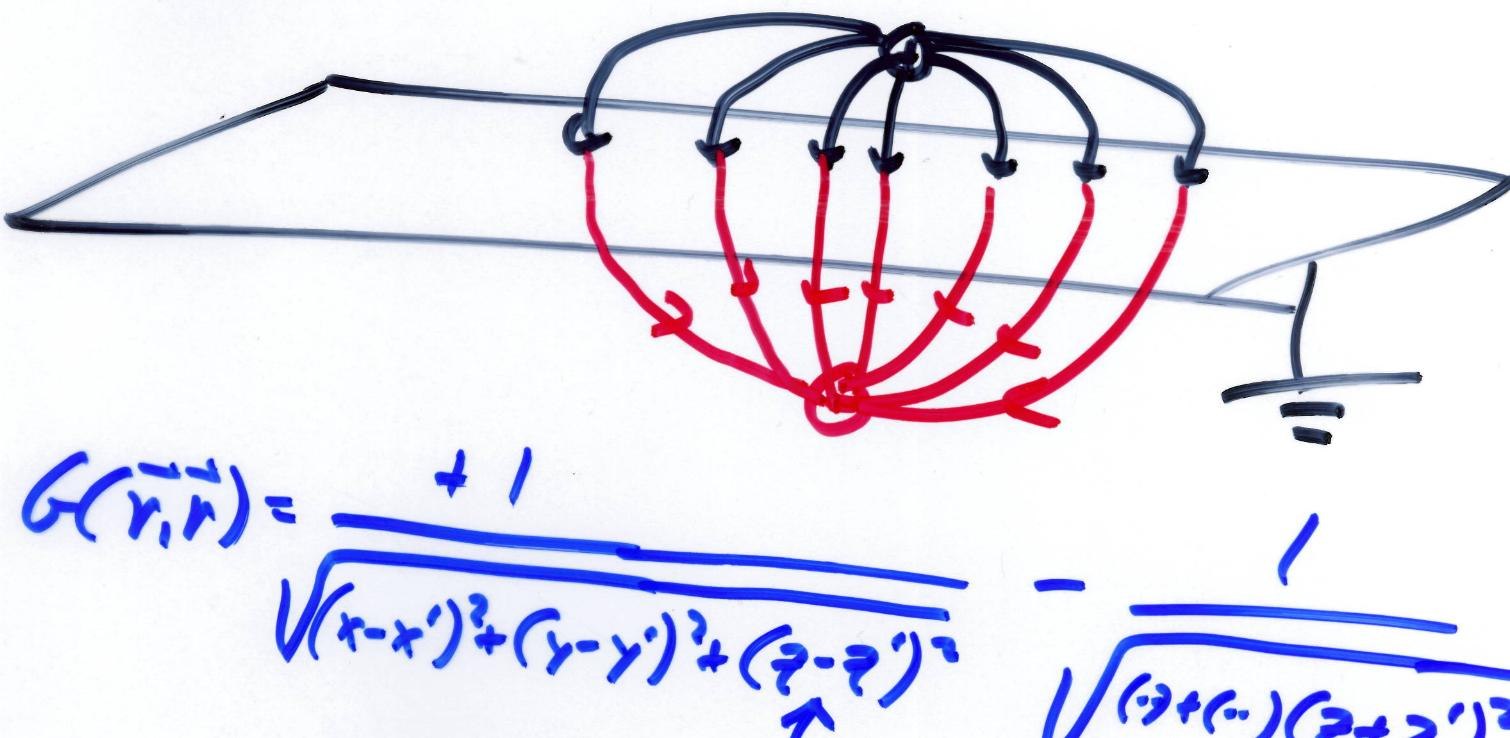
Laplace $\nabla^2 \Phi(\vec{r}) = 0$

2nd order, homogeneous linear partial diff. eq.

Poisson $\nabla^2 \Phi(\vec{r}) = -\dfrac{\rho(\vec{r})}{\epsilon_0}$

voltage

$$\nabla^2 G(\vec{r}, \vec{r}\,') = -4\pi\, \delta^3(\vec{r} - \vec{r}\,')$$

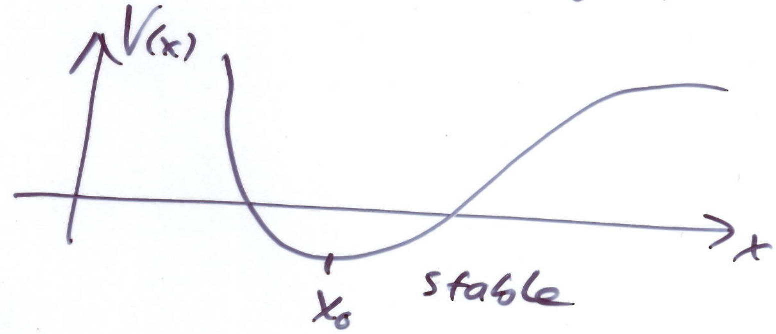


$$G(\vec{r}, \vec{r}\,') = \frac{+1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(\cdot) + (\cdot\cdot) + (z+z')^2}}$$

Small Oscillations (Coupled)

generalized coordinates : $\{q_1, q_2 \cdots q_N\}$

Stable Equilibrium



Potential Energy

$$V(q_1 \cdots q_N) = \overbrace{V(q_{10}, q_{20} \cdots q_{N0})}^{\text{ignore}} + \sum_{i=1}^{N} \cancelto{0}{\left.\frac{\partial V}{\partial q_i}\right|_0} \eta_i$$

$$+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \underbrace{\left.\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)\right|_0}_{V_{ij}} \eta_i \eta_j + \cdots$$

$q_i = q_{i0} + \eta_i$ ← small excursion from equilibrium

$\dot{q}_i = \dot{\eta}_i$

Kinetic Energy

$$T = \frac{1}{2}\sum_i \sum_j m_{ij} \dot{x}_i \dot{x}_j = \frac{1}{2}\sum_i \sum_j T_{ij} \dot{q}_i \dot{q}_j$$

depend on q_i's

$$L = T - V = \frac{1}{2}\sum_i \sum_j \left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j\right)$$

T_{ij}, V_{ij} are real symmetric matrices

Euler-Lagrange Equations of Motion

$$\frac{\partial L}{\partial \eta_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_k} = 0$$

N of these
$k \in \{1, N\}$

$$\frac{\partial L}{\partial \eta_k} = \frac{\partial}{\partial \eta_k}\frac{1}{2}\sum_i \sum_j \left(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j\right)$$

$$= -\frac{1}{2}\sum_i\sum_j V_{ij}\underbrace{\frac{\partial \eta_i}{\partial \eta_k}}_{\delta_{ik}}\eta_j - \frac{1}{2}\sum_i\sum_j V_{ij}\eta_i\underbrace{\frac{\partial \eta_j}{\partial \eta_k}}_{\delta_{jk}}$$

$$= -\frac{1}{2}\sum_j V_{kj}\eta_j - \frac{1}{2}\sum_i V_{ik}\eta_i$$

$$= -\sum_j V_{kj}\eta_j$$

(k is a free index
j is a dummy index)

$$\frac{\partial L}{\partial \dot{\eta}_k} = \sum_j T_{kj}\dot{\eta}_j$$

Euler-Lagrange: $\displaystyle\sum_{j=1}^{N}\left(T_{kj}\ddot{\eta}_j + V_{kj}\eta_j\right) = 0$

N coupled linear differential equations
– Solve simultaneously.

Ansatz: $\eta_i(t) = A_i e^{-i\omega t}$ ← Fourier time transform

A_i is complex

Euler-Lagrange: $\sum_j (V_{kj} - \omega^2 T_{kj}) A_j = 0$

$$\begin{bmatrix} (V_{11}-\omega^2 T_{11}) & (V_{12}-\omega^2 T_{12}) & \cdots & \\ V_{21}-\omega^2 T_{21} & \ddots & & \\ \vdots & & \ddots & \\ & & & V_{NN}-\omega^2 T_{NN} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{\underline{M}} \cdot \vec{A} = \vec{0}$$

If $\underline{\underline{M}}$ is invertible, then

$$\vec{A} = \underline{\underline{M}}^{-1} \cdot \underline{\underline{M}} \cdot \vec{A} = \underline{\underline{M}}^{-1} \cdot \vec{0} = \vec{0}$$ trivial solution

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So if $\vec{A} \neq \vec{0}$, then $\underline{\underline{M}}$ can not have an inverse $\Longrightarrow \det(\underline{\underline{M}}) = 0$