Physics Challenge for Teachers and Students

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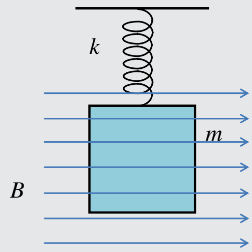
Spring vibes

A thin copper plate of mass m has a shape of a square with a side b and thickness d. The plate is suspended on a vertical spring with a force constant k in a uniform horizontal magnetic field B parallel to the plane of the plate. Find the period of the small-amplitude vertical oscillations of the plate.



We received a large number of solutions from all over the world for our January *Challenge*, **Hot, Cool, and Working Hard.** We are pleased to recognize the following authors of correct solutions:

Antonio Jorge Aranda Gomez (student, Escuela Superior Politecnica, Seville, Spain)
Sharmila Balamurugan (student, Women's Christian College, Chennai, India),
Hratch Barsoumian (Haigazian University, Beirut, Lebanon)
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Clint Sprott (University of Wisconsin – Madison, WI)

Many thanks to all contributors and we hope to hear from you in the future!

Guidelines for contributors:

- Email your solutions as Word files.
- If your name is — for instance —Bilbo Baggins, please name the file "April12Baggins" (**do not include your first initial**) if you are solving the April *Challenge*.
- State your name and professional affiliation in the file, not only in the email message.

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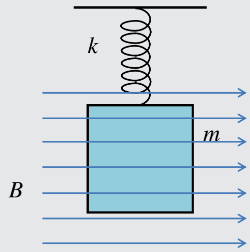
Solution to April 2012 *Challenge*

Spring vibes

A thin copper plate of mass m has a shape of a square with a side b and thickness d. The plate is suspended on a vertical spring with a force constant k in a uniform horizontal magnetic field B parallel to the plane of the plate. Find the period of the small-amplitude vertical oscillations of the plate.



Solution: The plate oscillates harmonically with displacement y from its equilibrium position as a function of time t,

$$y = A\sin(\omega t + \phi), \quad (1)$$

where the amplitude A and phase constant ϕ are determined by the initial conditions. Taking the first and second time derivatives of this expression gives the velocity

$$\upsilon = A\omega\cos(\omega t + \phi) \quad (2)$$

and the acceleration

$$a = -A\omega^2\sin(\omega t + \phi) \quad (3)$$

of the plate. The problem is to find the period T of oscillation, which is related to the angular frequency in the above formulae as $T = 2\pi/\omega$. In the absence of the magnetic field, that angular frequency would be simply $\omega = \sqrt{k/m}$. However in the presence of the field, there are induced voltages and currents that cost mechanical energy to establish. Those energies are associated with forces that in turn alter the frequency of oscillation.

Consider the plate at an instant that it is moving vertically downward with velocity υ. Viewed from its left edge, it is like a short but tall bar moving in a magnetic field. Its bottom edge is cutting across magnetic field lines, giving rise to a motional emf (or Hall voltage) $\varepsilon = B\upsilon d$. That voltage is created by a charge flow (i.e., eddy current) whose direction is out of the page in the given diagram according to Lenz's law. (Another way to see the direction is to imagine a small positive free charge moving downward with the plate. The magnetic force on that moving charge will push it toward the front face of the plate.) Consequently the front square face of the plate will build up a positive charge and the back face an equal and opposite negative charge. We thus have a parallel-plate capacitor with capacitance $C = \varepsilon_0 b^2/d$.

The energy stored in this capacitor is

$$U = \tfrac{1}{2}C\varepsilon^2 = \tfrac{1}{2}\varepsilon_0 b^2 B^2 \upsilon^2 d \quad (4)$$

after substituting in the preceding expressions for the capacitance and emf. The force associated with this potential energy is

$$\begin{aligned} F &= -\frac{dU}{dy} = -\varepsilon_0 b^2 B^2 \upsilon \frac{d\upsilon}{dy} d \\ &= -\varepsilon_0 b^2 B^2 \upsilon \frac{d\upsilon}{dt}\frac{dt}{dy} d = -\varepsilon_0 b^2 B^2 a d \end{aligned} \quad (5)$$

since $dy/dt = \upsilon$ and $d\upsilon/dt = a$. The minus sign indicates that this magnetic force is upward on the downward-moving plate, slowing down the oscillations.

Newton's second law for the plate now takes the form

$$ma = -ky - \varepsilon_0 b^2 B^2 a d \quad \Rightarrow \quad m_{\text{eff}}\, a = -ky, \quad (6)$$

where $m_{\text{eff}} = m + \varepsilon_0 b^2 B^2 d$. Note that it is not necessary to include the constant gravitational force mg if we measure y from the *equilibrium* not the *unloaded* position of the spring. We conclude that

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_{\text{eff}}}{k}} = \boxed{2\pi\sqrt{\frac{m + \varepsilon_0 b^2 B^2 d}{k}}}\,. \quad (7)$$

Three comments should be made about this solution. First, the correction to T due to the induced emf is extremely small. The ratio of the correction term to the uncorrected period is $\varepsilon_0 b^2 B^2 d/m = \varepsilon_0 B^2/\rho_{\text{m}}$, where the mass density of copper is ρ_m = 8960 kg/m^3. Even for a large magnetic field of B = 0.1 T, the fractional correction to the period is thus only 10^{-17}, which is negligible. For all practical purposes, the plate will oscillate with the same period as it would in the absence of the magnetic field! So this problem is purely of academic interest.

Second, Eq. (7) only reflects the contribution of the induced voltage. There is also a contribution due to the induced current. However we can show that it is smaller by yet another factor of 10^{-17} and hence is even more negligible. The magnitude of the charge on either of the square faces of the plate is $q = C\varepsilon = \varepsilon_0 b^2 Bv$. Thus the induced current is $I = dq/dt = \varepsilon_0 b^2 Ba$. This is associated with a power (Joule heating) of

$$P = I^2 R = \left(\varepsilon_0 b^2 Ba\right)^2 \frac{\rho d}{b^2}, \tag{8}$$

where R is the resistance of the plate in the direction perpendicular to the page, and ρ is the resistivity of copper. Substituting Eq. (3) into this expression, the average value of the power over a period of oscillation is

$$\begin{aligned} P_{\text{avg}} &= \varepsilon_0^2 b^2 B^2 \rho A^2 \omega^4 d \left\langle \sin^2(\omega t + \phi) \right\rangle , \\ &= \tfrac{1}{2} \varepsilon_0^2 b^2 B^2 \rho A^2 \omega^4 d \end{aligned} \tag{9}$$

and thus the average mechanical energy lost due to the eddy currents over a period is

$$E_{\text{avg}} = P_{\text{avg}} T = \pi \varepsilon_0^2 b^2 B^2 \rho A^2 \omega^3 d. \tag{10}$$

Compare this to the average of Eq. (4) over a period,

$$U_{\text{avg}} = \tfrac{1}{2} \varepsilon_0 b^2 B^2 A^2 \omega^2 d \left\langle \cos^2(\omega t + \phi) \right\rangle = \tfrac{1}{4} \varepsilon_0 b^2 B^2 A^2 \omega^2 d. \tag{11}$$

The ratio of Eq. (10) to (11) is

$$\frac{E_{\text{avg}}}{U_{\text{avg}}} = 8\pi^2 \frac{\varepsilon_0 \rho}{T} . \tag{12}$$

Ignoring the factor of $8\pi^2$, this result is simply the ratio of the time constant RC to the period of oscillation T. But ε_0 = 8.8 pF/m and the resistivity of copper is approximately 1.7×10^{-8} Ω·m, depending on its exact purity. Thus for realistic values of k and m (say, 100 N/m and 0.1 kg, respectively), Eq. (12) is equal to approximately 10^{-17}, which is negligible. Still, if one is going to include a term of fractional magnitude 10^{-17}, one may as well keep a term of fractional magnitude 10^{-34}.

(Contributed by Carl E. Mungan, U. S. Naval Academy, Annapolis, MD)

We would also like to recognize the following contributors:

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Clint Sprott (University of Wisconsin – Madison, WI)

Patrick van Nieuwenhuizen (retired, Harare, Zimbabwe)

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