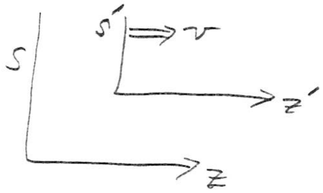
Ch 7

Lorentz Transformation Derived



Assume a linear relation:

$$z' = a_1 z + a_2 t$$
$$t' = a_3 t + a_4 z$$

Physical Laws Maintain their form in all inertial frames,
"Covariance"

The linear relation ensures Newton's First law is covariant.

$$\left.\begin{aligned} dz' &= a_1\,dz + a_2\,dt \\ dt' &= a_3\,dt + a_4\,dz \end{aligned}\right\} \quad \frac{dz'}{dt'} = \frac{a_1 \frac{dz}{dt} + a_2}{a_3 + a_4 \frac{dz}{dt}}$$

If $\frac{dz}{dt} = \text{const}$, then $\frac{dz'}{dt'} = \text{const}'$

Consider a point P fixed in S $\Rightarrow \Delta z = 0$

$$\Delta z' = a_2 \Delta t \qquad \frac{\Delta z'}{\Delta t'} = \frac{a_2}{a_3} = -v \qquad \boxed{a_2 = -v a_3}$$
$$\Delta t' = a_3 \Delta t$$

Consider a point P' fixed in S' $\Rightarrow \Delta z' = 0$

$$a_1 \Delta z + a_2 \Delta t = 0 \qquad \frac{\Delta z}{\Delta t} = -\frac{a_2}{a_1} = +v \qquad \boxed{a_2 = -v a_1}$$

$\Rightarrow a_1 = a_3$

We want the velocity of light to be the same in all frames
(Michelson Morley)

$$\frac{\Delta z'}{\Delta t'} = c \Longleftrightarrow \frac{\Delta z}{\Delta t} = c$$

We want $a_1(v)$ to be an <u>even</u> function of v

$$a_1(+v) = a_1(-v)$$

$$\Rightarrow [a_1(v)]^2 = \frac{1}{1-v^2/c^2} \Rightarrow a_1(v) = \frac{1}{\sqrt{1-v^2/c^2}} \equiv \gamma$$

Why?

① We want to preserve time-reversal invariance:

$z \to +z$

$t \to -t$

$v \to -v$ $\qquad a_1(v) \to +a_1(-v)$

$t' \to -t'$

② We want the <u>group</u> property:

$$S \xrightarrow[a_1(v)]{} S' \xrightarrow[a_1(-v)]{} ? \quad S$$

$$S \xrightarrow[a_1(v_1)]{} S' \xrightarrow[a_1(v_2)]{} S''$$

$$\underbrace{\qquad\qquad\qquad}_{a_1(v_3)}$$

$$S \xrightarrow[a_1(v_1)]{} S' \xrightarrow[a_1(v_2)]{} S''$$

$$\xleftarrow[a_1(-v_3)]{\qquad\qquad}$$

Two Lorentz transformations in succession give another Lorentz transformation.

$$a_1\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right) = a_1(v_1)\, a_1(v_2) \left[1+\frac{v_1 v_2}{c^2}\right]$$

$$\frac{d}{dv_2} a_1\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right) = a_1(v_1)\left[\frac{d}{dv_2} a_1(v_2)\left[1+\frac{v_1 v_2}{c^2}\right] + a_1(v_2)\frac{v_1}{c^2}\right]$$

$$a_1'\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right)\left[\frac{1}{1+\frac{v_1 v_2}{c^2}} - \frac{(v_1+v_2)\frac{v_1}{c^2}}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right] = a_1(v_1)\left[a_1'(v_2)\left(1+\frac{v_1 v_2}{c^2}\right) + a_1(v_2)\frac{v_1}{c^2}\right]$$

$\underline{v_2 = 0}$

$$a_1'(v_1)\left[1-\frac{v_1^2}{c^2}\right] = a_1(v_1)\left[a_1'(0) + a_1(0)\frac{v_1}{c^2}\right]$$

$a_1(0) = 1 \Rightarrow z = z'$ if no relative velocity

$$a_1'(v_1)\left(1-\frac{v_1^2}{c^2}\right) - a_1(v_1)\left[a_1'(0) + \frac{v_1}{c^2}\right] = 0$$

differential equation. Solution:

If $a_1'(0) = 0$ then $a_1(v_1) = \dfrac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} = \gamma$

$$a_1(v+\delta v) = a_1(v)\, a_1(\delta v)\left[1+\frac{v\,\delta v}{c^2}\right]$$

$$\frac{da_1(v)}{dv} = \lim_{\delta v \to 0} \frac{a_1(v+\delta v) - a_1(v)}{\delta v} = \lim_{\delta v \to 0} \frac{a_1(v)\left\{a_1(\delta v)\left(1+\frac{v\,\delta v}{c^2}\right) - 1\right\}}{\delta v}$$

$$= a_1(v)\frac{v}{c^2} \Rightarrow \left.\frac{da_1(v)}{dv}\right|_{v=0} = 0 = a_1'(0)$$