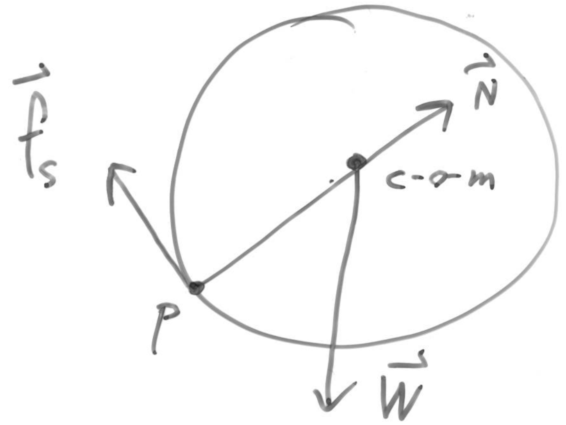
Origin O can be $\begin{cases} P \text{ - a point that does } \underline{\text{not}} \text{ accelerate.} \\ \text{or center-of-mass } \underline{\text{even}} \text{ if it acceleration.} \end{cases}$

Free-body diagram



$$\left.\begin{aligned} s &= R\theta \\ v &= R\omega \\ a &= R\alpha \end{aligned}\right] \text{ no slip}$$

$$I_{cm} = \frac{1}{2} m R^2$$

Steiner parallel Axis theorem

$$I_P = I_{cm} + mD^2$$

$$\Sigma \vec{\tau}_P = I_P \vec{\alpha}$$

$$f_s(0) + N(0) + mgR\sin\theta = \left[\frac{1}{2}mR^2 + mR^2\right]\frac{\vec{a}_{cm}}{R}$$

$$a_{cm} = \frac{2}{3} g \sin\theta$$

Get final speed from 1-dimensional kinematics

Constant acceleration

$$x_f = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$v_f = v_0 + at$$

$$\boxed{v_f^2 = v_0^2 + 2a(x_f - x_0)}$$

$$a(t) = a$$

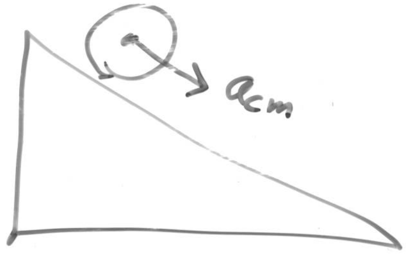
$$\Delta v = v_f - v_0 = \int a\, dt = at$$

$$\Delta x = x_f - x_0 = \int v(t)\, dt$$

$$= \int [at + v_0]\, dt$$

$v_f = \sqrt{\frac{4}{3} g h}$ rolling $<$ $v_f = \sqrt{2gh}$ sliding

Frictional force



$$\sum_{\substack{\text{along}\\ \text{ramp}}} F = m\, a_{cm}$$

$$mg\sin\theta - f_s = m a_{cm} = m \tfrac{2}{3} g \sin\theta$$

$$f_s = \tfrac{1}{3} mg \sin\theta$$

$$\sum_{\substack{\text{perpendicular}\\ \text{to ramp}}} F = m \cdot 0$$

$$N - mg\cos\theta = 0$$

$$N = mg\cos\theta$$

Method (2) Energy Conservation

Kinetic energy $T \equiv \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v}\cdot\vec{v}$

$$\frac{dT}{dt} = m\frac{d\vec{v}}{dt}\cdot\vec{v} = m\vec{a}\cdot\vec{v} = \vec{F}_{TOTAL}\cdot\vec{v} = \vec{F}_{TOTAL}\cdot\frac{d\vec{r}}{dt}$$

$$\int dT = \int \vec{F}_{tot}\cdot d\vec{r}$$

$\Delta T = T_f - T_i$ = Total work done on mass m by all the forces.

Work-Energy Theorem

T - kinetic energy $\qquad$ V - potential energy

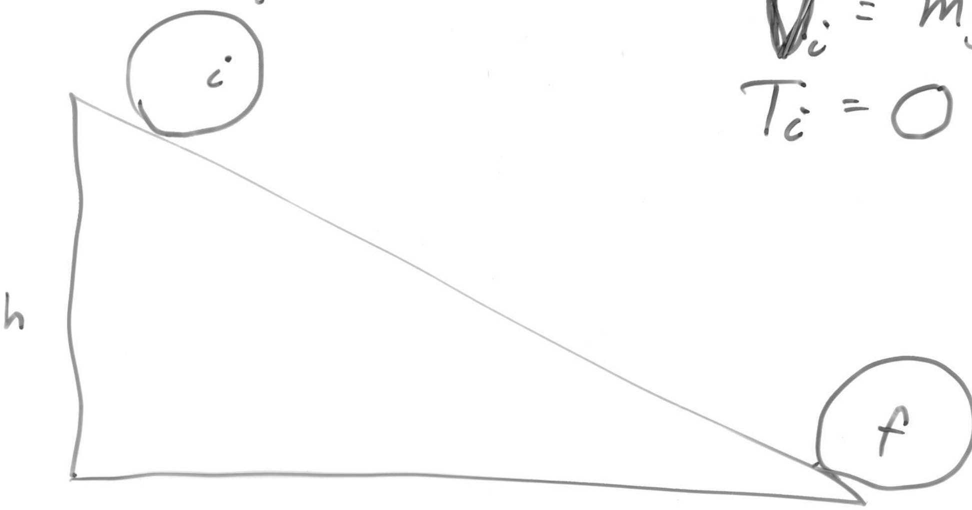
See if all the forces (3) are <u>conservative</u>.

$\Rightarrow$ Work done by the force depends only on the endpoints, not on the path.

$$\Rightarrow \vec{\nabla} \times \vec{F} = 0 \qquad \vec{\nabla} \times \vec{\nabla}(?) = 0$$

$$\Rightarrow \vec{F} = -\vec{\nabla} V$$

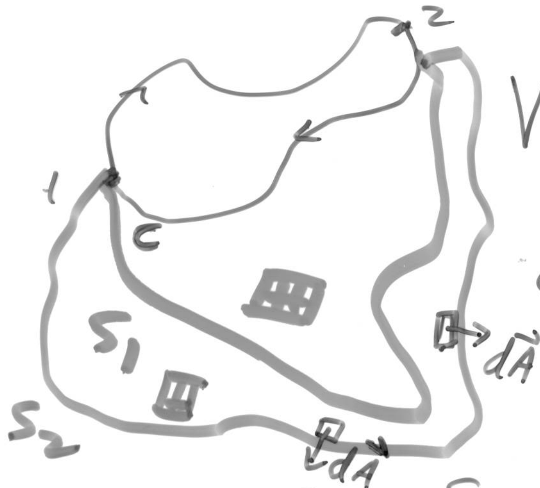
$$V_i = mgh$$
$$T_i = 0$$



$$V_f = 0$$
$$T_f = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$V_i + T_i = V_f + T_f$$
$$mgh + 0 = 0 + \frac{1}{2} m v_{cm}^2 + \frac{1}{2}\left[\frac{1}{2} m R^2\right]\left(\frac{v_{cm}}{R}\right)^2$$

$\vec{F}$ has zero curl $\Longrightarrow$ work is path independent



$$W = \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}$$

closed

open

open surface S has the closed curve C as its boundary. C has no boundary. The boundary of the boundary is zero.

Differential Forms

exterior derivative

$$\int_{\partial M} \omega = \int_M d\omega$$

$$\boxed{\partial^2 M = 0}$$

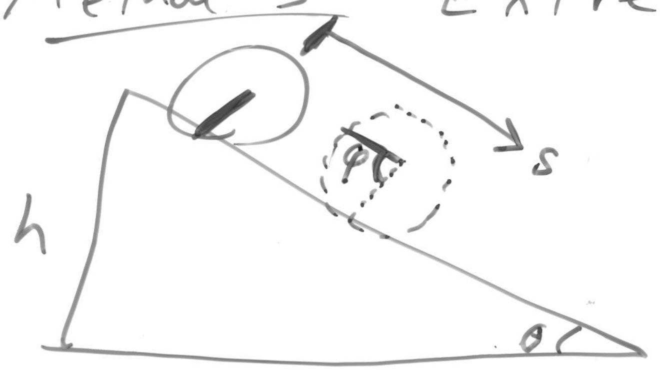
M - manifold (surface)

∂M - boundary of M

Divergence Theorem (Gauss' Theorem)

$$\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV$$

Method 3   Extremization Principle



Lagrangian $L = T - V$

Action $I = \int L\, dt$

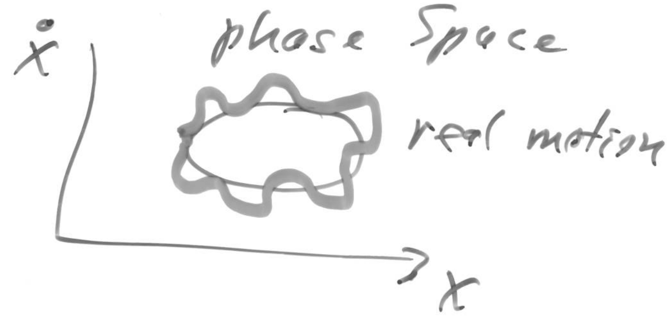
Extermize I

$\boxed{\dot{s} = v_{cm}}$

$\boxed{\dot{\varphi} = \omega}$

$$L = L(x_i, \dot{x}_i; t) \qquad i = \{1, 2\}$$

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$



$$T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} I_{cm} \dot{\varphi}^2$$

$$V = -mgs \sin\theta$$

$$L = \frac{1}{2} m \dot{s}^2 + \frac{1}{4} m R^2 \dot{\varphi}^2 + mgs \sin\theta$$

Constraint: roll without slipping: $s - R\varphi = 0$

$f = s - R\varphi = 0$ holonomic, schleronomous

$x_i = \{s, \varphi\}$

Lagrange multiplier

$$\frac{\partial L}{\partial s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) + \lambda \frac{\partial f}{\partial s} = 0$$

$$\Longrightarrow mg\sin\theta - m\ddot{s} + \lambda = 0 \qquad (A)$$

$$\frac{\partial L}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) + \lambda \frac{\partial f}{\partial \varphi} = 0$$

$$-\frac{1}{2}mR^2\ddot{\varphi} - \lambda R = 0 \qquad (B)$$

$$\Longrightarrow \lambda = -\frac{1}{2}mR\ddot{\varphi} = -\frac{1}{2}m\ddot{s} \quad \text{using } f$$

substitute λ into (A)

$$mg\sin\theta - m\ddot{s} - \frac{1}{2}m\ddot{s} = 0 \longrightarrow \ddot{s} = \frac{2}{3}g\sin\theta$$

$$= a_{cm}$$

angular acceleration

$$\alpha = \ddot{\varphi} = \frac{-2\lambda}{mR} = -\frac{2}{mR}\left[-\frac{m}{2}\ddot{s}\right] = \frac{2}{3}\frac{g\sin\theta}{R}$$

Both simple 2^{nd} order differential equation

$\longrightarrow$ integrate twice $\longrightarrow$ $\dot{s}, s, \dot{\varphi}, \varphi$

Generalized "forces" of constraint.

force $\rightarrow Q_s = \lambda \frac{\partial f}{\partial s} = \lambda = -\frac{1}{3}mg\sin\theta \longleftarrow f_s$

torque $\rightarrow Q_\varphi = \lambda \frac{\partial f}{\partial \varphi} = -\lambda R = -\frac{1}{3}mgR\sin\theta = f_s R = \tau_s$