Conservation Laws

For a single particle, Newton's 2^{nd} Law

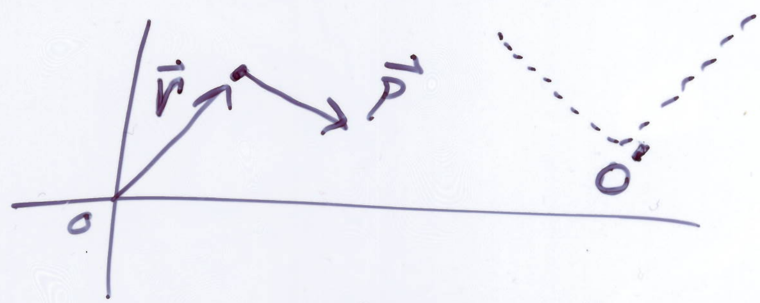
$$\vec{F}_{TOT} = \frac{d\vec{p}}{dt}$$

If the total force is zero, then the linear momentum of the particle is conserved (is constant in time)

$$\vec{F}_{TOT} = 0 \implies \frac{d\vec{p}}{dt} = 0 \implies \vec{p} = \text{constant} \implies \vec{p}(t_1) = \vec{p}(t_2).$$

Angular Momentum about origin O

$$\vec{L}_O \equiv \vec{r} \times \vec{p}$$



Torque $\vec{N}_O^{(1)} = \vec{r} \times \vec{F}^{(1)}$, $\vec{N}_O^{(2)} = \vec{r} \times \vec{F}^{(2)}$, ...

$$\vec{N}_{O\,TOT} = \vec{r} \times \vec{F}_{TOT} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times m\frac{d\vec{v}}{dt}$$

If m = constant.

Notice : $\frac{d}{dt}(\vec{r} \times \vec{v}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{v}}_{\vec{v} \times \vec{v} = 0} + \vec{r} \times \frac{d\vec{v}}{dt}$

$$\vec{N}_{\underset{o}{TOT}} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{L}_o}{dt} = \dot{\vec{L}}_o$$

If the total torque acting a particle is zero, then the angular momentum of that particle is conserved.

If all the forces acting on a particle are conservative, then

$$\vec{\nabla} \times \vec{F}_{TOT} = 0 \Longrightarrow \oint \vec{F}_{TOT} \cdot d\vec{r} = 0 \Longrightarrow \vec{F}_{TOT} = -\vec{\nabla} V(\vec{r})$$

stokes' theorem

total derivative

$$\vec{F}_{TOT} \cdot d\vec{r} = -(\vec{\nabla} V) \cdot d\vec{r} = -dV$$

Work: $W = \int_i^f \vec{F}_{TOT} \cdot d\vec{r} = \int_i^f m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \frac{m}{2} \int_i^f \frac{d}{dt}(v^2)\, dt$

$$= \frac{m}{2}(v_f^2 - v_i^2) = T_f - T_i$$

$\leftarrow$ kinetic energy

$$W = -\int_i^f dV = -V_f + V_i$$

$$-V_f + V_i = T_f - T_i \Longrightarrow V_i + T_i = V_f + T_f$$

total mechanical energy = kinetic + potential.

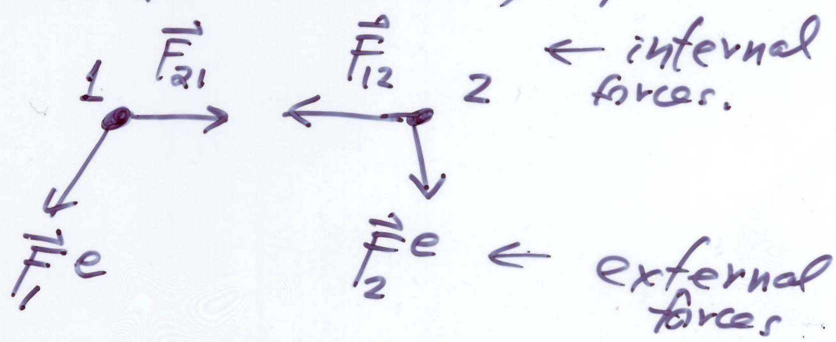
If the forces acting on a particle are conservative, then the total mechanical energy of the particle is conserved.

Note: $V = V(\vec{r})$ only coordinates

If $V(\vec{r}, t)$ **, then** $T + V$ **not conserved.**

Consider a system of N interacting particles.

First, just two



Internal forces are caused by some particle in the system and act on a different particle in the system.

External forces are caused by some agent not in the system.

Newton's 3rd law

$$\vec{F}_{12} = -\vec{F}_{21}$$

Weak form of Law of action and reaction

This is not always true! E.g. velocity-dependent forces $\vec{F}_{mag} = q\,\vec{v} \times \vec{B}$

Total force on the i^{th} particle.

$$\vec{F}_{\substack{tot\\i}} = \vec{F}_i^{\,e} + \sum_{j=1}^{N}{}' \vec{F}_{ji} = \dot{\vec{P}}_i$$

prime on sum means omit $j=i$ term

Total forces on the system

$$\sum_{i=1}^{N} \vec{F}_{\substack{tot\\i}} = \sum_i \vec{F}_i^{\,e} + \sum_{i,j}{}' \vec{F}_{ji} = \frac{d^2}{dt^2} \sum_i m_i \vec{r}_i$$

total external force on system $\vec{F}_{tot}^{\,e}$

Zero, forces will cancel in pairs if Weak form 3rd Law holds.

Define the center of mass position

$$\vec{R}_{cm} \equiv \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M_{tot}}$$

$$\Longrightarrow \vec{F}_{Tot}^{\,e} = M_{tot} \frac{d^2 \vec{R}_{cm}}{dt^2}$$

The center of mass (c-o-m) of a system moves as if all the mass were concentrated there and subject to the net external forces.

E.g. Exploding Bomb

The total Linear momentum of the system

$$\vec{P}_{Tot} = \sum_i \vec{P}_i = \sum_i m_i \frac{d\vec{r}_i}{dt} = M_{Tot} \frac{d\vec{R}_{cm}}{dt} \equiv M_{Tot} \vec{V}_{cm}$$

$$\vec{F}^e_{Tot} = \dot{\vec{P}}_{Tot}$$

If the total external force on a system is zero, then the total linear momentum of the system is conserved.

$$\vec{F}^e_{Tot} = 0 \implies \vec{P}_{Tot} = \overrightarrow{constant} \implies \vec{P}_{tot}(t_1) = \vec{P}_{tot}(t_2)$$

Total Angular Momentum of a system

$$\vec{L}_{\underset{o}{Tot}} = \sum_i \vec{L}_{\underset{o}{i}} = \sum_i \vec{r}_i \times m_i \vec{v}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\dot{\vec{L}}_{o\,TOT} = \sum_i \frac{d}{dt}(\vec{r}_i \times m_i \vec{v}_i)$$

$$= \underbrace{\sum_i \vec{v}_i \times m \vec{v}_i}_{\text{zero}} + \sum_i \vec{r}_i \times \dot{\vec{P}}_i = \sum_i \vec{r}_i \times \vec{F}_{i\,TOT}$$

$$= \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^e} + \underline{\sum_{i,j}{}' \vec{r}_i \times \vec{F}_{ji}} \quad ?$$

$$\sum_i \vec{N}_{oi}^e = \vec{N}_{o\,TOT}^e$$

= total external torque about origin O.

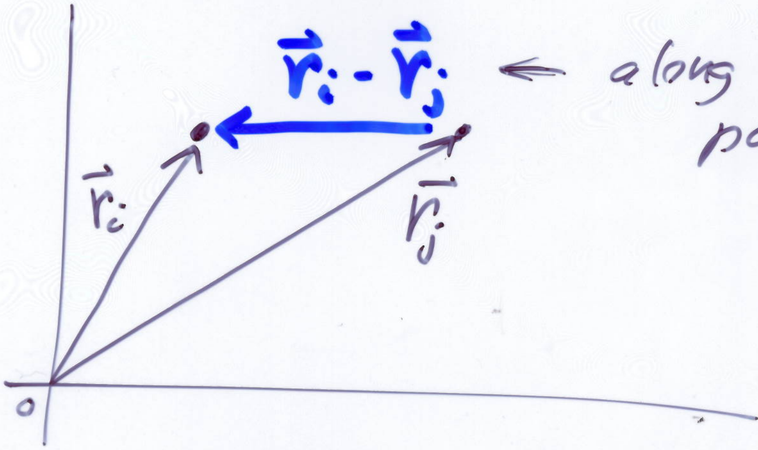
$$\sum_{i,j}{}' \vec{r}_i \times \vec{F}_{ji} = \sum_{\substack{i=1 \\ (i \neq j)}}^{N} \sum_{j=1}^{N} \vec{r}_i \times \vec{F}_{ji} = \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \sum_{i=1}^{N} \vec{r}_j \times \vec{F}_{ij}$$

$$= \frac{1}{2} \sum_{i,j}{}' (\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij}).$$

$$= \frac{1}{2} \sum_{i,j}{}' (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$$

using weak form of 3rd Law

that is, $\vec{F}_{ij} = -\vec{F}_{ji}$



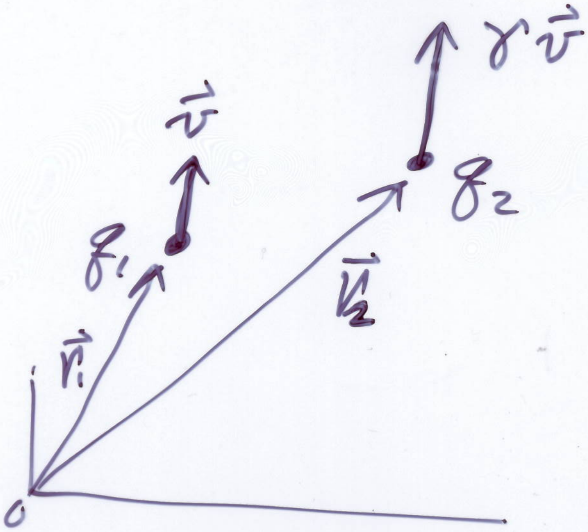
← along the line joining the particles.

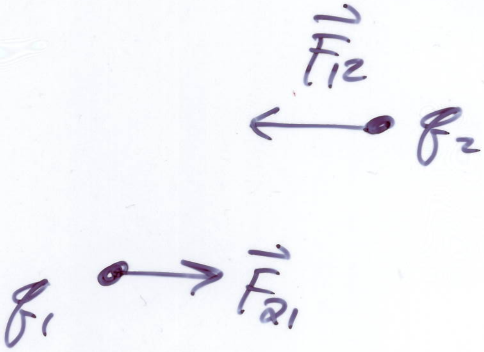
So $\frac{1}{2} \sum_{i,j}{}' (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$ will be zero if the forces also act on a line joing the particles (E.g. Coulomb electric force, gravity)

This is called the strong form of the 3rd Law.

$$\vec{N}^{e}_{\substack{tot\\0}} = \dot{\vec{L}}_{\substack{tot\\0}}$$

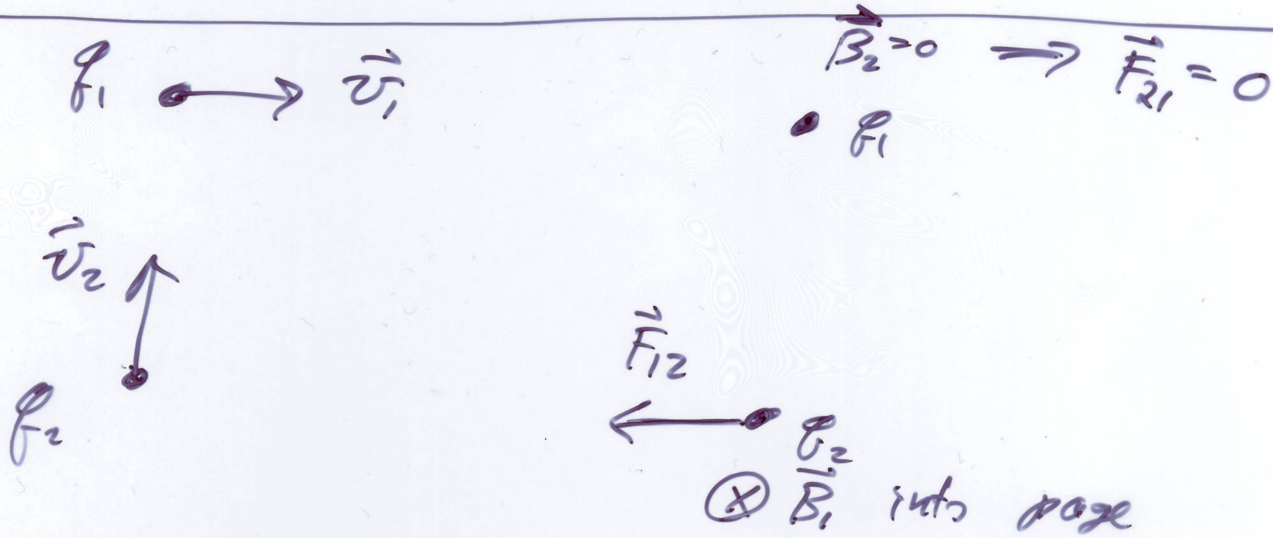
If the total external torque is zero, then the total angular momentum of the system is constant. **(If the strong form of the 3rd law holds.**





satisfies the weak form but violates the strong form of the thiRD Law

$$\vec{F}_{12} = -\vec{F}_{21}$$



Here even the weak form of the 3rd Law is violated

$$\vec{F}_{12} \neq -\vec{F}_{21}$$