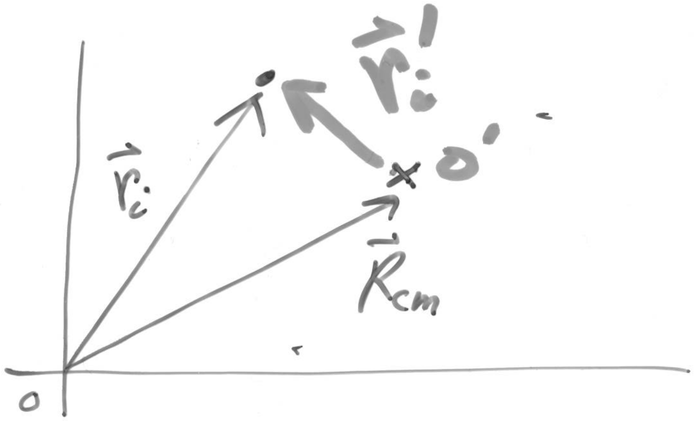
We want a relation analogous to

$$\vec{P}_{TOT} = M_{TOT} \frac{d\vec{R}_{cm}}{dt} = M_{TOT} \vec{V}_{cm}$$

but for angular momentum.



$$\vec{r}_i = \vec{R}_{cm} + \vec{r}_i'$$

$$\vec{v}_i = \vec{V}_{cm} + \vec{v}_i'$$

$$\vec{L}_{O\,TOT} = \sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

$$= \sum_i (\vec{R}_{cm} + \vec{r}_i') \times m_i (\vec{V}_{cm} + \vec{v}_i')$$

radius vector of c-o-m in the prime system.

$$= \sum_i m_i \vec{R}_{cm} \times \vec{V}_{cm} + \sum_i \vec{r}_i' \times m_i \vec{v}_i' + \overbrace{\sum_i m_i \vec{r}_i'}^{0} \times \vec{V}_{cm}$$

$$+ \sum_i \vec{R}_{cm} \times \underbrace{m_i \vec{v}_i'}_{\frac{d}{dt}\sum_i m_i \vec{r}_i' = \frac{d}{dt} 0}$$

$$\vec{L}_{O\,tot} = \vec{R}_{cm} \times M_{TOT} \vec{V}_{cm} + \sum_i \vec{r}_i' \times \vec{P}_i'$$

the total angular momentum about origin O is the angular momentum of the center of mass (all mass concentrated ~~there~~) plus the angular momentum about the center of mass.

e.g.

$$\vec{L}_{\substack{tot \\ sun}} = \text{orbital } L + \text{spin } L$$

eg. electron in a state not s-state

$$\vec{J} = \underset{\substack{\uparrow \\ \text{orbital}}}{\vec{L}} + \underset{\substack{\nwarrow \\ \text{spin}}}{\vec{S}}$$

Kinetic energy of a system of particles

$$T_{tot} = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \sum_i \tfrac{1}{2} m_i \left(\vec{V}_{cm} + \vec{v}_i'\right) \cdot \left(\vec{V}_{cm} + \vec{v}_i'\right)$$

$$= \sum_i \tfrac{1}{2} m_i \vec{V}_{cm}^2 + \sum_i \tfrac{1}{2} m_i v_i'^2 + \underbrace{\sum_i m_i \vec{v}_i'}_{\frac{d}{dt} \underbrace{\sum_i m_i \vec{r}_i'}_{0}} \cdot \vec{V}_{cm}$$

$$= \underset{\substack{\uparrow \\ \text{Translation of} \\ \text{center of mass}}}{\tfrac{1}{2} M_{Tot} \vec{V}_{cm}^2} + \underset{\substack{\nwarrow \\ \text{Rotation about} \\ \text{center of mass}}}{\sum_i \tfrac{1}{2} m_i v_i'^2}$$

Constraints

explicit dependence

holonomic : $h(\vec{r}_1, \vec{r}_2, \dots, t) = 0$

with time t : rheonomous

no explicit t : scleronomous $h(\vec{r}_1, \vec{r}_2 \dots \vec{r}_N) = 0$

e.g. spherical pendulum

constraint

$$h(x, y, z) = h(\vec{r}_1) = x^2 + y^2 + z^2 - l^2 = 0$$

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non-holonomic:

e.g.

constraint

$$x^2 + y^2 + z^2 - a^2 \leq 0$$

e.g. A condition of derivatives of the coordinates that is <u>not</u> integrable.

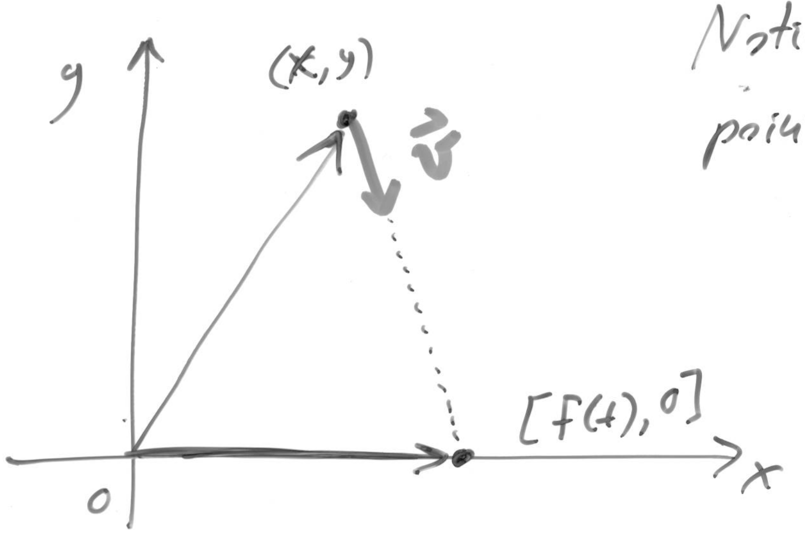
$$\frac{dx}{dt} + 3\frac{dy}{dt} = 0 \implies dx = -3dy \implies x = -3y + C$$

$$h(x,y) = x + 3y - C = 0$$

this is holonomic.

Derivation 1.6

A particle moves in the xy plane such that its velocity vector always points toward the point $[f(t), 0]$



Notice $f(t)\hat{e}_x - (x\hat{e}_x + y\hat{e}_y)$ points from (x,y) to $[f(t), 0]$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y = A\left[f(t)\hat{e}_x - (x\hat{e}_x + y\hat{e}_y)\right]$$

$$\dot{x} = A[f(t) - x] \qquad \dot{y} = -Ay$$

$$\frac{\dot{x}}{\dot{y}} = \frac{dx/dt}{dy/dt} = \frac{dx}{dy} = \frac{x - f(t)}{y} \Rightarrow \frac{dx}{x - f(t)} - \frac{dy}{y} = 0$$

This is non-holonomic — two ways.

Method 1:

$$dx \frac{\partial}{\partial x} \ln[x - f(t)] - dy \frac{\partial}{\partial y} \ln[y] = 0$$

$$\left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right) \ln\left[\frac{x - f(t)}{y}\right] = 0$$

ALmost total derivative

$$\left(d - dt \frac{\partial}{\partial t}\right) \ln\left[\frac{x - f(t)}{y}\right] = 0$$

$$d = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dt \frac{\partial}{\partial t}$$

$$d \ln\left[\frac{x - f(t)}{y}\right] = \frac{-1}{x - f(t)} \dot{f}\, dt$$

If $\dot{f} = 0$ (f = constant), then this would be $d\, h(x, y, t) = 0$

$$h(x, y, t) = \ln\left[\frac{x - f(t)}{y}\right] + C = 0$$

Is there a constraint $h(x, y, t) = 0$ for our problem?

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial t} dt = 0$$

$\Rightarrow$ dy and dt can be chosen independently and together they determine dx.

but in our problem dy alone determines dx.

therefore $\frac{\partial h}{\partial t} = 0$

$$\frac{dx}{x - f(t)} = \frac{dy}{y} \qquad \Bigg| \qquad \frac{\partial h}{\partial x} dx = -\frac{\partial h}{\partial y} dy$$

$$-\frac{\partial h / \partial x}{\partial h / \partial y} = \frac{y}{x - f(t)}$$

t dependence

no t dependence

contradiction

Method 2

$$y\,dx - [x - f(t)]\,dy = 0$$

Look for an integrating factor $g(x, y, t)$ that will make the constraint a total derivative of $h(x,y,t) = 0$.

$$dh = 0 = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial t}dt = 0$$

$$g(x,y,t)\,y\,dx - g(x,y,t)[x - f(t)]\,dy = 0$$

$$\frac{\partial h}{\partial x} = g(x,y,t)\,y \quad \Big| \quad \frac{\partial h}{\partial y} = -g(x,y,t)[x - f(t)] \quad \Big| \quad \frac{\partial h}{\partial t} = 0$$

$$\frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial x}\frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t}\frac{\partial h}{\partial x} = 0 \Rightarrow$$

$$\frac{\partial}{\partial t}[g(x,y,t)\,y] = 0 \Rightarrow \underset{\text{not time}}{g(x,y)}$$

$$\frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial y}\frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t}\frac{\partial h}{\partial y} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\{-g(x,y)[x - f(t)]\} = 0 \Rightarrow -g(x,y)\dot{f} = 0$$

If $g(x,y)\,\mathring{f} = 0$ then

either $\dot{f} = 0$ so $f =$ constant

or $g = 0 \rightarrow$ no integrating factor

$\rightarrow$ there is no $h(x,y,t) = 0$

so the original constraint

$y\,dx - [x - f(t)]\,dy = 0$ is ~~non~~-holonomic

can't be integrated.

Goldstein page 15.

eq. 1.39 a) $dx - a \sin\theta \, d\varphi = 0$

Look for an integrating factor $g(x,y,\theta,\varphi)$ such that

$$g(x,y,\theta,\varphi)\,dx - a\,g(x,y,\theta,\varphi)\sin\theta\, d\varphi = 0$$

is the total derivative of a holonomic constraint $dh(x,y,\theta,\varphi) = 0$

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial \theta}d\theta + \frac{\partial h}{\partial \varphi}d\varphi = 0$$

$$\frac{\partial h}{\partial x} = g(x,y,\theta,\varphi) \;\Big|\; \frac{\partial h}{\partial y} = 0 = \frac{\partial h}{\partial \theta} \;\Big/\; \frac{\partial h}{\partial \varphi} = -a\,g(x,y,\theta,\varphi)\sin\theta$$

$$\frac{\partial h}{\partial \theta} = 0 \Rightarrow \frac{\partial}{\partial x}\frac{\partial h}{\partial \theta} = 0 \Rightarrow \frac{\partial}{\partial \theta}\frac{\partial h}{\partial x} = 0 \Rightarrow \frac{\partial}{\partial \theta} g = 0$$

g does not depend on θ

$g(x, y, \varphi)$

$$\frac{\partial h}{\partial \theta} = 0 \Rightarrow \frac{\partial}{\partial \varphi}\frac{\partial h}{\partial \theta} = 0 \Rightarrow \frac{\partial}{\partial \theta}\frac{\partial h}{\partial \varphi} = 0$$

$$\Rightarrow \frac{\partial}{\partial \theta}\left[-a\, g(x, y, \varphi) \sin\theta\right] = 0$$

$$\Rightarrow -a\, g(x, y, \varphi) \cos\theta = 0$$

$\Rightarrow g = 0$ there is no integrating factor

$\Rightarrow$ there is no holonomic constraint h such that $dh = 0$