Systems of particles

N particles $\Longrightarrow$ $3N$ coordinates

with k constraints $\Longrightarrow$ $(3N-k)$ degrees of freedom

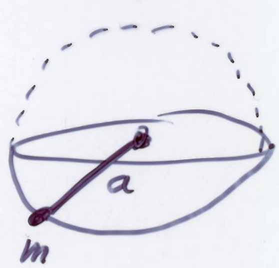
If the constraints are holonomic

$$h(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N ; t) = 0 \checkmark$$

$$dh = 0 \checkmark$$

then you can choose $(3N-k)$ generalized coordinates q_i consistent with the constraints and independent. These need not be lengths like $\vec{r}_j$; they could be angles, also energy, angular momentum, even Fourier coefficients.

e.g. Spherical Pendulum



$$\vec{r}_1(q_1, q_2 \cdots, q_{3N-k}, t)$$
$$\vec{r}_2(q_1, q_2 \cdots, q_{3N-k}, t)$$
$$\vdots$$
$$\vec{r}_N(\quad \cdots \quad)$$

$$h(x,y,z,t) = 0$$
$$x^2+y^2+z^2-a^2=0$$
holonomic

$$x = a \sin\theta \cos\varphi$$
$$y = a \sin\theta \sin\varphi$$
$$z = a \cos\theta$$

$$q_1 = \theta$$
$$q_2 = \varphi$$

d'Alembert's Principle

Split forces into forces of constraint $\vec{f}^c$ and applied forces $\vec{F}^a$. Consider virtual displacements $\delta\vec{r}_i$ of the system consistent with the constraints, at a given time t.

<u>Not</u> actual displacement $d\vec{r}_i$ that occurs in time dt during which the forces of constraint may be changing.

Newton's 2nd Law $\vec{F}_i = \dot{\vec{p}}_i \Rightarrow \vec{F}_i - \dot{\vec{p}}_i = 0$

$$\boxed{\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0}$$

$$\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i + \cancelto{0}{\sum_i \vec{f}_i^c \cdot \delta\vec{r}_i} = 0$$

not useful because $\vec{r}_1 \cdots \vec{r}_N$ not independent.
Introduce generalized coordinates $\{q_i, t\}$
Those will be independent

$$\vec{r}_i(q_1, q_2, \cdots, q_{3N-k}, t)$$

$\vdots$

Velocity

$$\vec{v}_i = \dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j} \frac{dq_j}{dt}$$

$$= \frac{\partial \vec{r}_i}{\partial t} + \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$$

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\frac{\partial \vec{r}_i}{\partial t} + \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right] = 0 + \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta_{jk}$$

$$= \frac{\partial \vec{r}_i}{\partial q_k}$$

cancelling the dots

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$$

Virtual displacements

$$\delta \vec{r}_i = \sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

no time dependence

Virtual Work

$$\sum_{i=1}^{N} \vec{F}_i^a \cdot \delta \vec{r}_i = \sum_j \sum_i \vec{F}_i^a \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \equiv \sum_j Q_j \delta q_j$$

Generalized "Force" $Q_j = \sum_{i=1}^{N} \vec{F}_i^a \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

free index

summed index = dummy index

$$\sum_{i=1}^{N} \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_j \left[\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right] \delta q_j$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$ integrate by parts

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$$= \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right]$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial^2 \vec{r}_i}{\partial t\, \partial q_j} + \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_k\, \partial q_j} \dot{q}_k$$

$$= \frac{\partial}{\partial q_j} \left[\frac{\partial \vec{r}_i}{\partial t} + \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right]$$

$$= \frac{\partial}{\partial q_j} \frac{d \vec{r}_i}{dt} = \frac{\partial \vec{v}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{v}_i}{\partial \dot{q}_j}$$ previous page