$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

replace $\dot{\vec{r}}_i \rightarrow \vec{v}_i$

$$= \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right]$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \tfrac{1}{2} m_i v_i^2 \right) - \frac{\partial}{\partial q_j}\left(\sum_i \tfrac{1}{2} m_i v_i^2 \right)$$

$$= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$$

$$\sum_{i=1}^{3N} \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{j=1}^{3N-k} \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

$$\sum_i \vec{F}_i^{a} \cdot \delta \vec{r}_i = \sum_j Q_j \, \delta q_j$$

$$0 = \sum_i \left(\dot{\vec{p}}_i - \vec{F}_i^{a} \right) \cdot \delta \vec{r}_i$$

$$= \sum_j \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

$$\Rightarrow \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad \forall_j$$

If the forces are conservative,

then $\vec{F}_i^a = -\vec{\nabla}_i V$

$$Q_j = \sum_i \vec{F}_i^a \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i (\vec{\nabla}_i V) \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}(T-V) = 0$$

If the potential energy V does <u>not</u> depend on the generalized velocities,

then $\frac{\partial V}{\partial \dot{q}_j} = 0 \quad \forall j$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial q_j}(T-V) = 0$$

Lagrangian $L \equiv T - V$

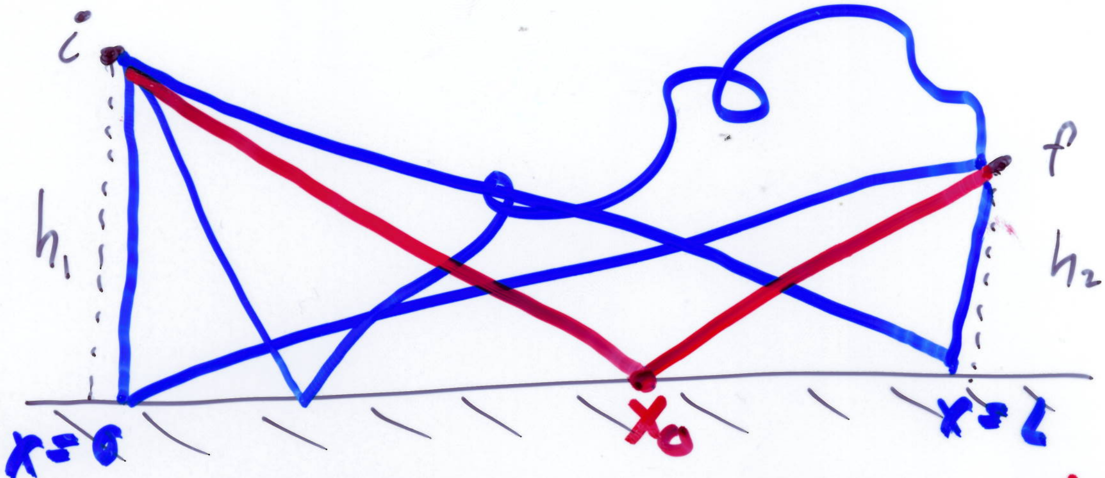
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Euler-Lagrange Eq.
equations of motion
$(3N-k)$ of them.

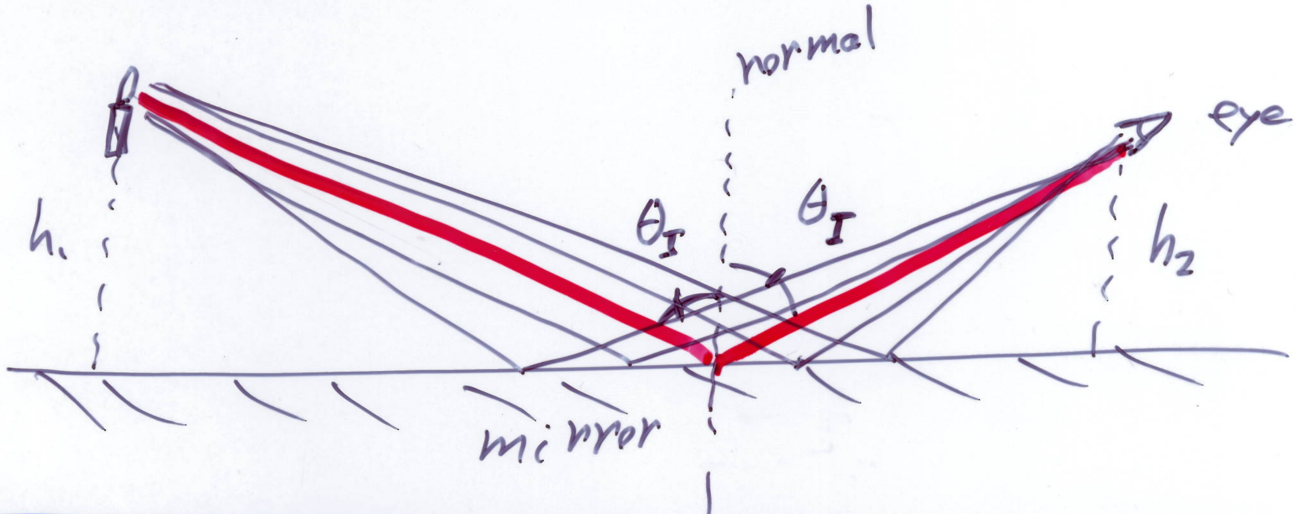
Extremization Principles

e.g. Specular Reflection by Fermat's principle of Least Time

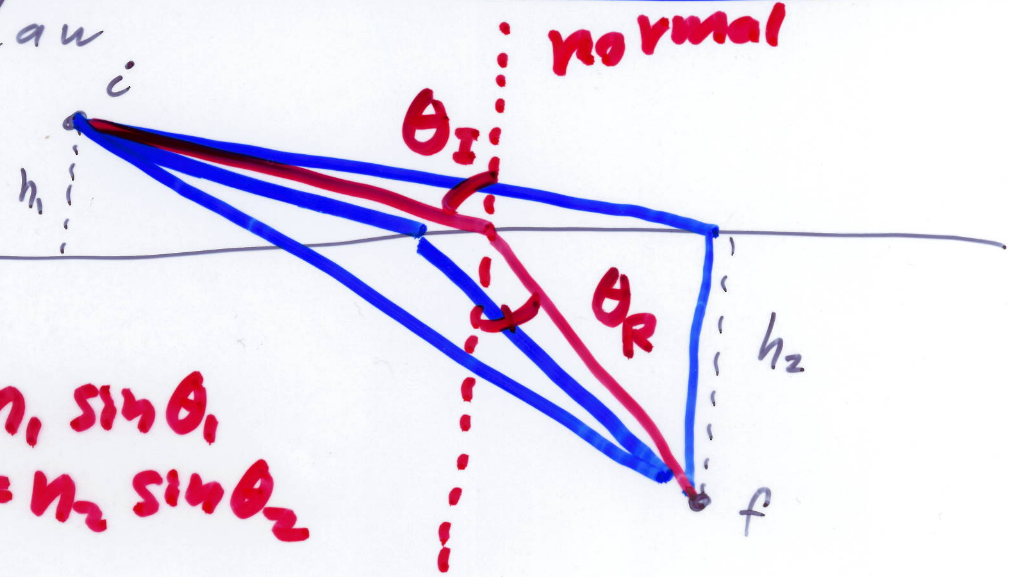
speed v



red path minimizes time



e.g. Snell's Law



speed $= v$

sand

water

speed $\frac{v}{n}$

$n \geq 1$

$n_1 \sin\theta_1 = n_2 \sin\theta_2$

If $\rho(\vec{r})$ is the charge density and $\Phi(\vec{r})$ is the electrostatic potential (voltage)

Poisson's Eq: $$\nabla^2 \Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

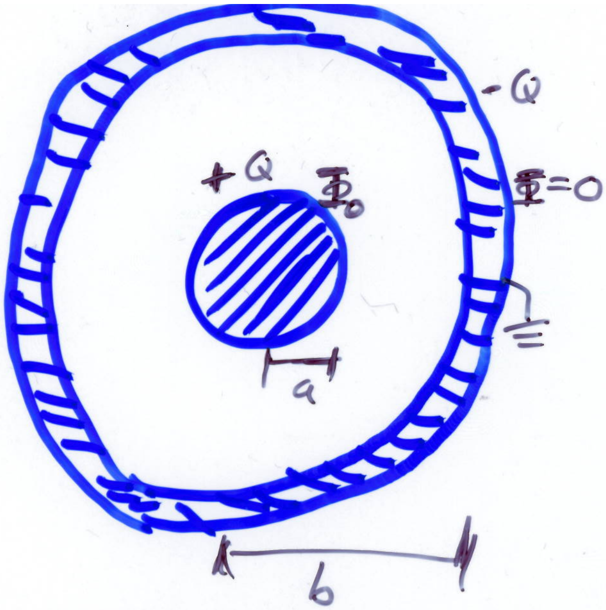
2nd order, linear, non-homogeneous partial differential equation.

Mechanics: We can get $\vec{F} = m\vec{a}$ from $L = T - V$.

$$U^* = \frac{\epsilon_0}{2} \underbrace{\iiint}_{\text{All space}} \Big[\underbrace{(\vec{\nabla}\Phi)\cdot(\vec{\nabla}\Phi)}_{\sim \text{Kinetic term}} - \underbrace{\rho\Phi}_{\text{potential energy}} \Big] dV$$

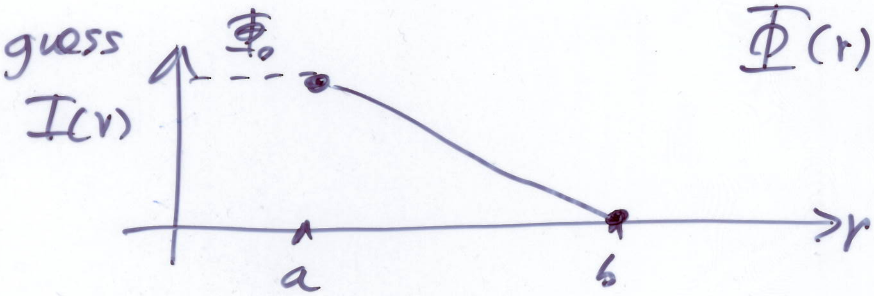
Look for a function $\Phi(\vec{r})$ that minimizes U^*

e.g. Consider the space between two charged metal cylinders.



$$\rho(r) = 0 , \quad a < r < b$$

$$U^* = \frac{\epsilon_0}{2} \underbrace{\iiint}_{\text{between plates}} (\vec{\nabla}\Phi)^2 \, dV$$



$$\Phi(r) = \Phi_0 \left(1 - \frac{r-a}{b-a}\right)$$

$$U^* = \frac{\epsilon_0}{2} \iiint \vec{E}(\vec{r}) \cdot \vec{E}(\vec{r}) \, dV = \text{energy in field}$$

$$= \frac{1}{2} C \Phi_0^2 \leftarrow \text{for capacitor.}$$

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guess #2