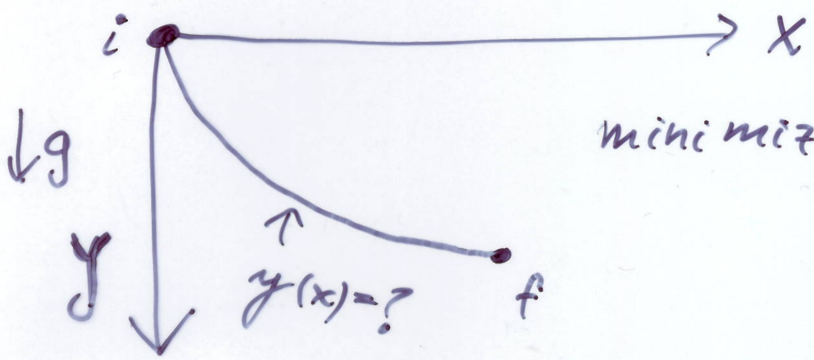
Beltrami form for $L(q, \dot{q}, t)$

$$-\frac{\partial L}{\partial t} + \frac{d}{dt}\left[L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right] = 0$$

useful if L does not depend explicitly on time.

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \text{constant}$$

Back to the Brachistochrone



minimize travel time

$$f[y(x), \dot{y}(x), x] = \sqrt{\frac{1 + \dot{y}^2}{2gy}}$$

$\frac{\partial f}{\partial x} = 0$ no explicit x dependence

Beltrami form: $f - \dot{y}\frac{\partial f}{\partial \dot{y}} = \text{const.}$

$$\sqrt{\frac{1+\dot{y}^2}{2gy}} - \frac{\dot{y}^2}{\sqrt{1+\dot{y}^2}}\frac{1}{\sqrt{2gy}} = \text{const.}$$

$$\frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = \text{const.}$$ square both sides

$$\frac{1}{(2g\,y)(1+\dot{y}^2)} = \text{const.}$$

$$y\,(1+\dot{y})^2 \equiv a = y\left[1+\left(\frac{dy}{dx}\right)\right]^2$$

$$\frac{dy}{dx} = \sqrt{\frac{a}{y}-1}$$

parametric solution

$$x(\theta) = \tfrac{1}{2}a\,(\theta - \sin\theta) + b$$

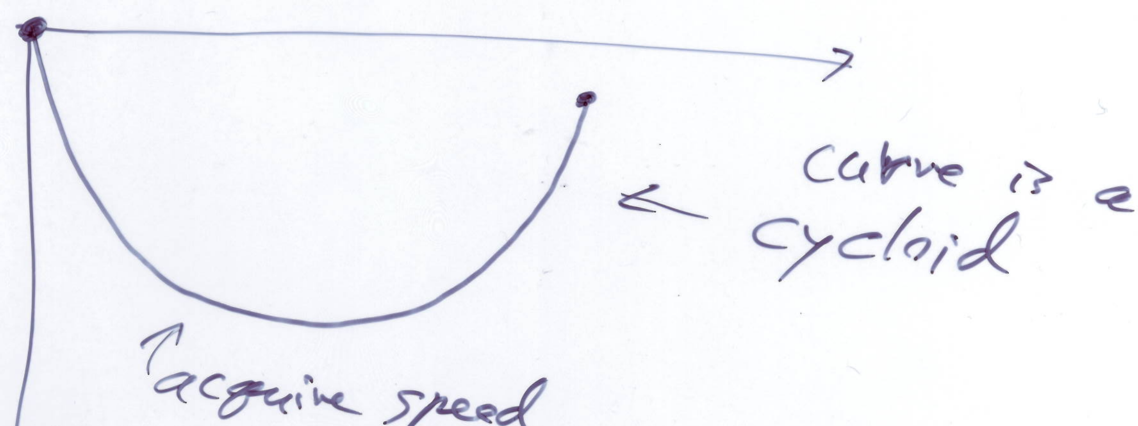
$$y(\theta) = \tfrac{1}{2}a\,(1-\cos\theta)$$

$$\frac{dx}{d\theta} = \tfrac{1}{2}a\,(1-\cos\theta)$$

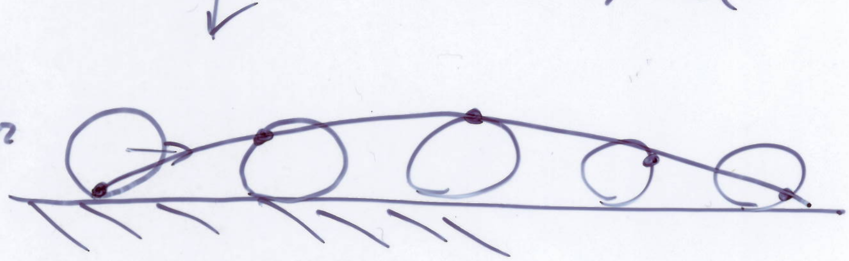
$$\frac{dy}{d\theta} = \tfrac{1}{2}a\,\sin\theta$$

$$\dot{y} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

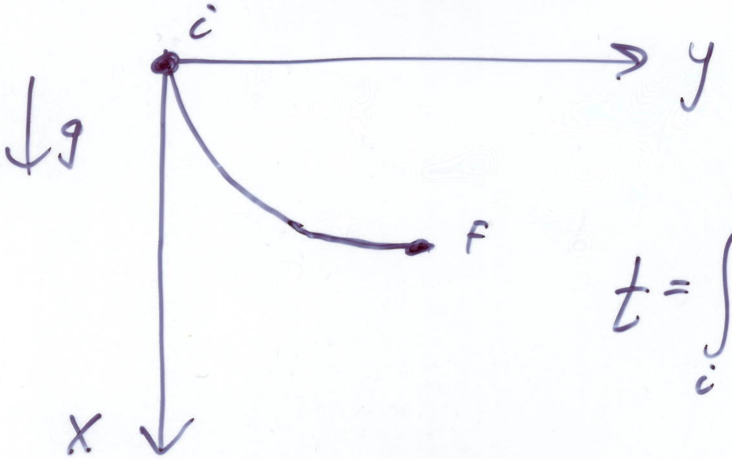
example



rolling cylinder



Third method - choose (x,y) differently



$$v = \sqrt{2gx}$$

$$t = \int_i^f \frac{ds}{v} = \int_{x_i}^{x_f} \sqrt{\frac{1+\dot{y}^2}{2gx}}\, dx$$

$$f[y(x), \dot{y}(x), x] = \sqrt{\frac{1+\dot{y}^2}{2gx}}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y \text{ is cyclic}$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} = 0 \Rightarrow \frac{\partial f}{\partial \dot{y}} = \text{const.}$$

$$\Rightarrow \frac{1}{\sqrt{2gx}} \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = \text{const.}$$

$$\dot{y} = \frac{dy}{dx} = \frac{x}{\sqrt{ax - x^2}}$$

$$y = \int dy = \int \frac{x\, dx}{\sqrt{ax - x^2}}$$

change variables
$x = \frac{1}{2}a(1-\cos\theta)$

$$= \frac{a}{2}\int (1-\cos\theta)\, d\theta = \frac{a}{2}(\theta - \sin\theta) + \text{const.}$$

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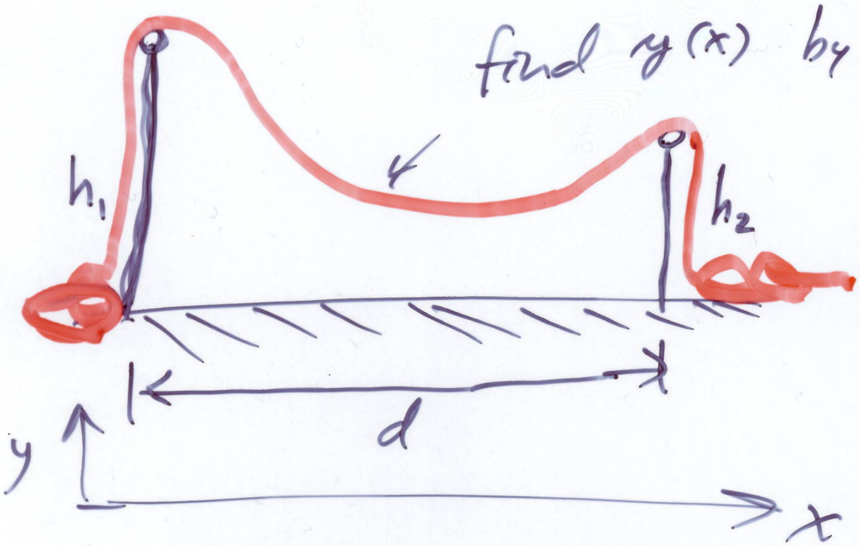
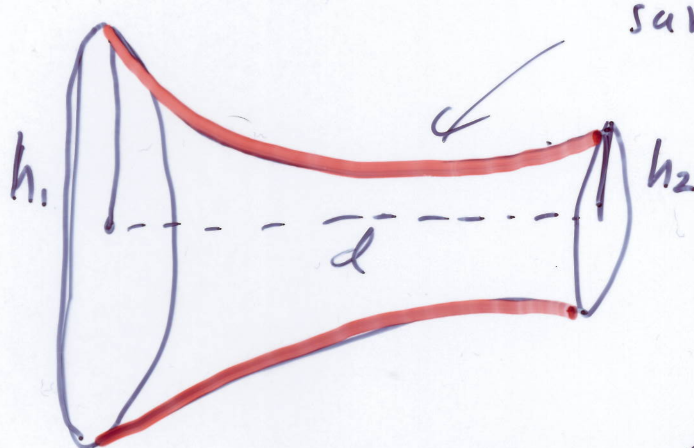
Catenary (Latin for chain is catena)



find $y(x)$ by minimizing potential energy.

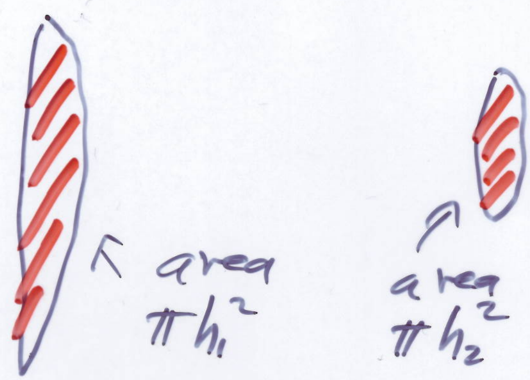
chain has uniform mass density μ.



surface of revolution has minimal surface area.

Soap film will adopt this shape.

If d is too large (Goldstein Ch2-7,8)



Goldschmidt solution

Galileo in <u>The Two Sciences</u> mentions a parabolic approximation to the curve. Joachim Jungius proved the curve was not a parabola. published posthumously in 1669.

Suspension Bridge

"massless" $\ll m_{deck}$

$$U = \int g y \, dm = \int g y \mu \, ds = g\mu \int_{x=0}^{d} y\sqrt{1+\dot{y}^2}\, dx$$

$$f[y(x), \dot{y}(x), x] = y\sqrt{1+\dot{y}^2}$$

Beltrami form : $\frac{\partial f}{\partial x} = 0$

$$\Rightarrow f - \dot{y}\frac{\partial f}{\partial \dot{y}} = \text{constant}$$

first constant of integration.

$$\frac{\partial f}{\partial \dot{y}} = \frac{y\dot{y}}{\sqrt{1+\dot{y}^2}} \Rightarrow y\sqrt{1+\dot{y}^2} - \frac{y\dot{y}^2}{\sqrt{1+\dot{y}^2}} = \text{const}$$

$$\Rightarrow \frac{y}{\sqrt{1+\dot{y}^2}} = \text{const} = a \rightarrow \frac{y^2}{1+\dot{y}^2} = a^2$$

$$\frac{dy}{dx} = \dot{y} = \sqrt{\frac{y^2-a^2}{a^2}} \Rightarrow \int \frac{a\,dy}{\sqrt{y^2-a^2}} = \int dx = x$$

$$x = a \operatorname{arccosh}\left(\frac{y}{a}\right) + b \Rightarrow y = a\cosh\left(\frac{x-b}{a}\right)$$