Hamilton's Principle with constraints

$$I = \int_{t_1}^{t_2} \left(L + \sum_{j=1}^{k} \lambda_j f_j \right) dt$$

for k holonomic constraints

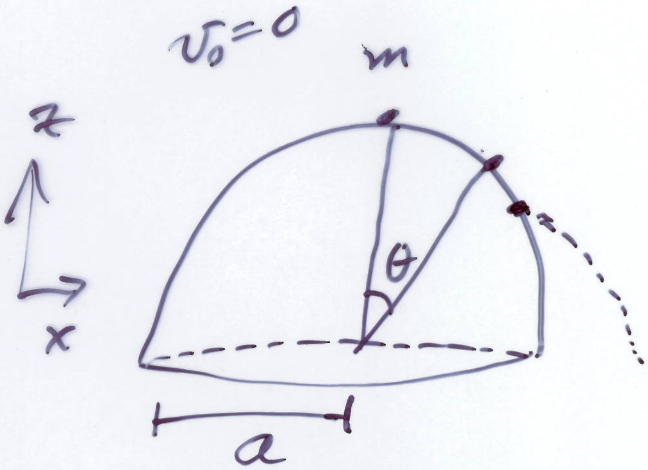
$$f(q_i) = 0$$

$$0 = \delta I = \int_{t_1}^{t_2} \sum_{i=1}^{3N} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i} \right) \delta q_i \, dt$$

but the $3N$ virtual displacements δq_i are not independent. $3N-k$ of the δq_i are independent. Choose the k λ_j's independently.

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \underbrace{\sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_j}}_{Q_i} = 0$$

Q_i generalized "forces" that produce the constraints.



$$T = \frac{m}{2}(\dot{x}^2 + \dot{z}^2)$$

$$V = mgz$$

$$f = \sqrt{x^2+z^2} - a = 0$$

Pick good generalized coordinates: r, θ

Switch to spherical coordinates

$$L' = T - V + \lambda f$$
$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta + \lambda(r-a)$$

$$\frac{\partial L'}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda$$

$$\frac{\partial L'}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L'}{\partial \theta} = mgr\sin\theta$$

$$\frac{\partial L'}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{r}} = \frac{\partial L'}{\partial r} \implies m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda$$

Use constraint $r = a$, $\dot{r} = 0$, $\ddot{r} = 0$

$$\implies 0 = ma\dot{\theta}^2 - mg\cos\theta + \lambda$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{\theta}} - \frac{\partial L'}{\partial \theta} = 0 \implies ma^2\ddot{\theta} - mga\sin\theta = 0$$

$$a\ddot{\theta} = g\sin\theta$$ multiply both sides by $\dot{\theta}$

$$a\ddot{\theta}\dot{\theta} = g\sin\theta\,\dot{\theta}$$

$$\frac{a}{2}\frac{d}{dt}\left(\dot{\theta}^2\right) = -\frac{d}{dt}\left(g\cos\theta\right) \implies \dot{\theta}^2 = -\frac{2g}{a}\cos\theta + C$$

fix C with initial condition $\dot{\theta} = 0$ when $\theta = 0$

$$\implies C = \frac{2g}{a}$$

$$\dot{\theta}^2 = -\frac{2g}{a}\cos\theta + \frac{2g}{a}$$

Substitute into Euler-Lagrange eq. for r

$$0 = ma\left[-\frac{2g}{a}\cos\theta + \frac{2g}{a}\right] - mg\cos\theta + \lambda$$

$$\Rightarrow \lambda = mg(3\cos\theta - 2)$$

The generalized "force" is $Q = \lambda \frac{\partial f}{\partial r} = \lambda$

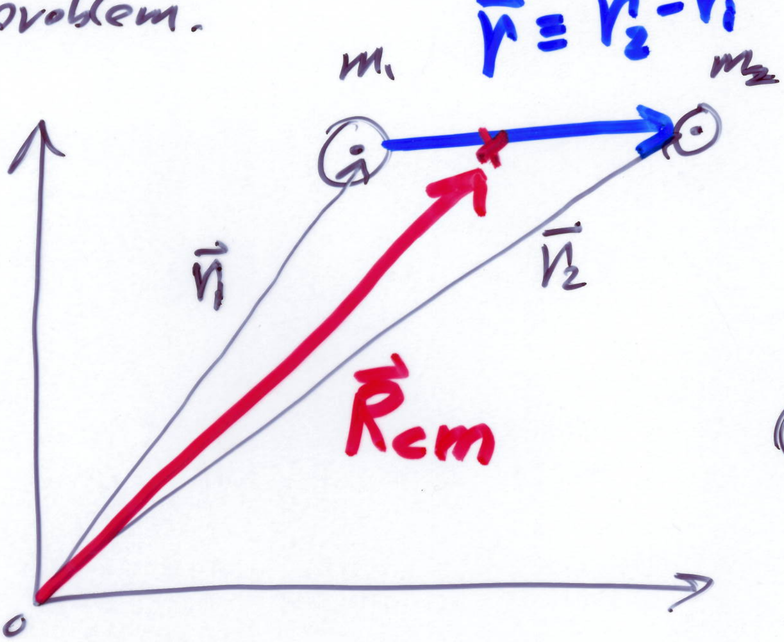
This is the normal force.

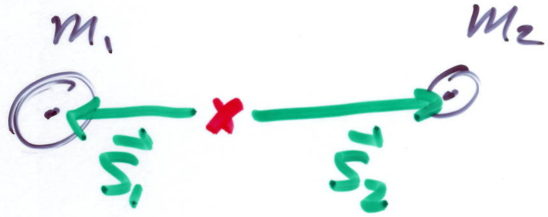
Mass m leaves the sphere when $Q = 0$

$$\lambda = 0 \Rightarrow 3\cos\theta = 2 \Rightarrow \theta_{crit} = \arccos\left(\frac{2}{3}\right) = 48.2^\circ$$

Central Forces

Reduce the two body problem to a one body problem.





Kinetic energy

$T =$ (T of the center of mass) + T about the center of mass

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}_{cm}^{\,2} + \frac{m_1}{2}\dot{\vec{s}}_1^{\,2} + \frac{m_2}{2}\dot{\vec{s}}_2^{\,2}$$

Two conditions on $\vec{s}_1$ and $\vec{s}_2$

$$\vec{s}_2 - \vec{s}_1 = \vec{r} \qquad m_1\vec{s}_1 + m_2\vec{s}_2 = 0$$

$$\Longrightarrow \quad \vec{s}_1 = \frac{-m_2}{m_1+m_2}\vec{r} \qquad \vec{s}_2 = \frac{m_1}{m_1+m_2}\vec{r}$$

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2}\mu\dot{\vec{r}}^{\,2}$$

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

Reduced mass

$$\mu < m_1, m_2 \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots$$

Aside: $\vec{r}^{\,2} = \vec{r}\cdot\vec{r} = r^2 = |\vec{r}|^2$

$$\dot{\vec{r}}^{\,2} \neq \dot{r}^2$$

$$\dot{\vec{r}}^{\,2} = \dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = \vec{v}\cdot\vec{v} = v^2$$

$$\dot{r}^2 = \left(\frac{dr}{dt}\right)^2 = \left(\frac{d|\vec{r}|}{dt}\right)^2$$

In Cartesian Coordinates

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\dot{\vec{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$$

$$\dot{\vec{r}}^{\,2} = v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\dot{r} = \frac{dr}{dt} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2+y^2+z^2}}$$

$$(\dot{r})^2 = \frac{(x\dot{x} + y\dot{y} + z\dot{z})^2}{x^2+y^2+z^2}$$

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A central force depends only on the distance between the particles, and lies along the line joining the particles.

If the force is conservative, then the Potential Energy function is $V(r)$

$$\vec{F} = -\vec{\nabla} V(r) = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}\right)$$

on Earth: $\hat{r}$ up, $\hat{\theta}$ south, $\hat{\varphi}$ east

$$= -\frac{dV}{dr}\hat{r}$$ only in radial direction

$\Rightarrow$ Spherical symmetry $\Rightarrow$ no dependence on the angles θ or φ

$\Rightarrow$ θ, φ are cyclic. $\Rightarrow$ momenta p_θ + p_φ will be conserved.

$\Rightarrow$ angular momentum vector (3-components) will be conserved. $\vec{\ell}$ = constant

$\vec{\ell}$ is called a first integral of the motion.

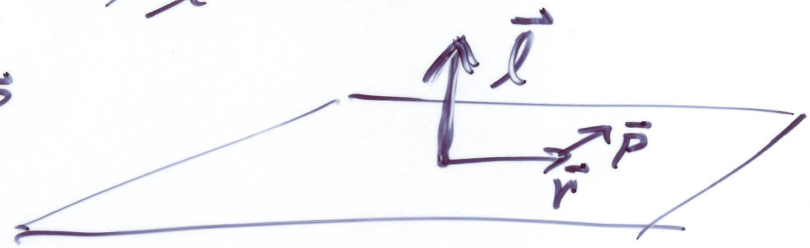
E is another first integral of the motion.

Another demonstration that $\vec{l}$ = constant

Torque $(\vec{\tau})$ $\vec{N} = \vec{r} \times \vec{F} \propto \vec{r} \times \vec{r} = 0 = \frac{d\vec{l}}{dt}$

$\Rightarrow \vec{l}$ = constant

$\vec{l} = \vec{r} \times \vec{p}$



Choose the $\hat{z}$ axis along $\vec{l}$

Motion happens in equatorial plane.

Use θ instead of φ as the azimuthal angle $0 \leq \varphi \leq 2\pi$ in this chapter.

$$L = T - V = \frac{1}{2}\mu(\underbrace{\dot{r}^2}_{v_r^2} + \underbrace{r^2\dot{\theta}^2}_{v_\theta^2}) - V(r)$$

$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ = constant $= l$

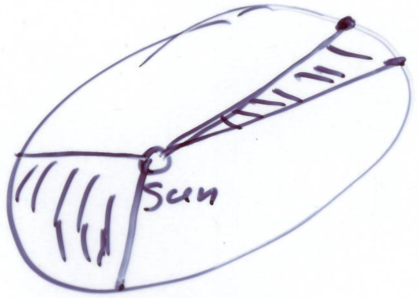
angular momentum

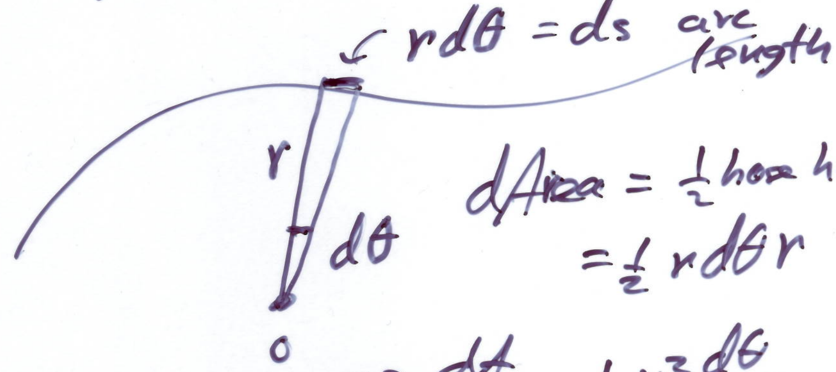
$l = \mu r^2 \dot{\theta}$

$\frac{dl}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0 \Rightarrow$ Areal velocity $\frac{1}{2}r^2\dot{\theta}$ = constant

Kepler's 2^{nd} Law

equal areas in equal times





$$dArea = \tfrac{1}{2}\,\text{base height}$$
$$= \tfrac{1}{2}\, r\,d\theta\, r$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Consequence of central force, not $\frac{1}{r^2}$.

Still holds for $V(r) = \frac{1}{r^4}$, r^2, ...