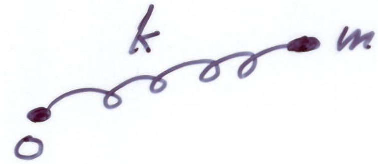
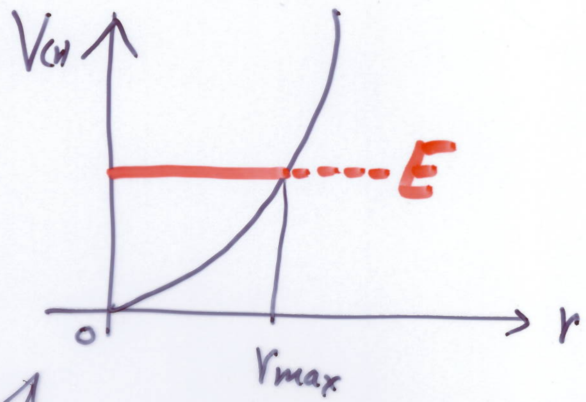
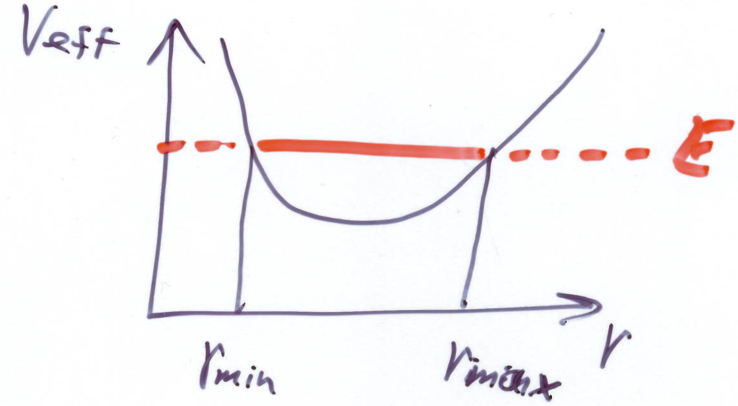
Ideal Hooke's Law Spring



$$\vec{f}(r) = -k\vec{r} \qquad V(r) = \tfrac{1}{2}kr^2$$





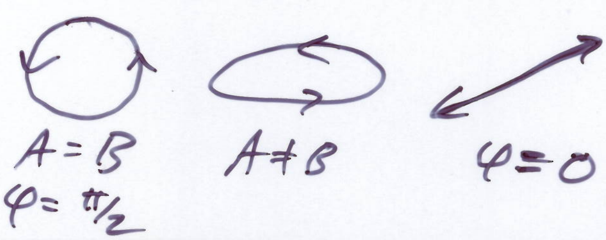
also V_{eff} if $\ell = 0$

straight line through origin

$$\vec{f} = -k\vec{r} \implies f_x = -kx \qquad f_y = -ky$$

two orthogonal independent simple harmonic oscillators – same frequency, different phases.

$$x = A\cos(\omega t)$$
$$y = B\cos(\omega t + \varphi)$$



Virial theorem

Newton 2nd: $\dot{\vec{P}}_i = \vec{F}_i$

$$G \equiv \sum_{i=1}^{N} \vec{P}_i \cdot \vec{r}_i$$

$$\frac{dG}{dt} = \sum_i \dot{\vec{P}}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \dot{\vec{r}}_i$$

$$\downarrow \qquad \hookrightarrow \sum m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \sum_i m_i v_i^2 = 2T$$

$$\sum_i \vec{F}_i \cdot \vec{r}_i$$

Time average $Z(t)$

$$\langle Z \rangle = \frac{1}{\tau}\int_{t=0}^{\tau} Z(t)\,dt$$

$$\left\langle \frac{dG}{dt} \right\rangle = \frac{1}{\tau}\int_{t=0}^{\tau} \frac{dG}{dt}\,dt = \frac{1}{\tau}\left[G(\tau)-G(0)\right] = 0$$

① G is periodic, choose $\tau = n$ periods

② $\vec{p}_i, \vec{r}_i$ bounded, let $\tau \to \infty$

$$= 2\langle T \rangle + \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle$$

$$\Rightarrow \langle T \rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle$$

Virial – Rudolf Clausius

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e.g. Ideal Gas Law

Equipartion theorem $\Rightarrow$ each atom has $\frac{1}{2}k_B T$ per degree of freedom. $\langle T \rangle = \frac{3}{2}k_B T N$

ideal gas – ignore interatomic forces. only forces of constraint from walls.

$$\sum_i \vec{F}_i \cdot \vec{r}_i = -P \oint\!\!\oint_{\text{walls}} \vec{r} \cdot d\vec{A} = -P \iiint (\vec{\nabla} \cdot \vec{r})\,dV = -P3V$$

$$\vec{\nabla} \cdot \vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = 3$$

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle \Rightarrow \frac{3}{2}N k_B T = \frac{3PV}{2} \Rightarrow PV = N k_B T$$

If $\vec{F}$ is central and comes from a potential energy

$$V(r) \rightarrow \vec{F} = -\vec{\nabla} V \Rightarrow |F| = \frac{\partial V}{\partial r}$$

$$\langle T \rangle = +\frac{1}{2}\left\langle r \frac{\partial V}{\partial r} \right\rangle$$

If $V = k r^{n+1}$ so $F \propto r^n$ then $r\frac{\partial V}{\partial r} = (n+1) V$

$$\langle T \rangle = \frac{n+1}{2}\langle V \rangle \Rightarrow$$

- Inverse square law force $n = -2$

  $\langle T \rangle = \frac{-1}{2}\langle V \rangle$
- Hooke's law $n = +1$

  $\langle T \rangle = \langle V \rangle$

Last time, we found $r(t)$ and $\theta(t)$ parametrized by time t. Often we want to know the <u>shape</u> of the orbit: $r(\theta)$

$$l = m r^2 \dot{\theta} = m r^2 \frac{d\theta}{dt} \longrightarrow dt = \frac{m r^2}{l} d\theta$$

$$\frac{d}{dt} = \frac{l}{m r^2}\frac{d}{d\theta}$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - \underbrace{V(r) - \frac{l^2}{2mr^2}}_{V_{eff}}\right)}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}} \Rightarrow d\theta = \frac{l}{m r^2}\frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$$

Convenient Change of Variable

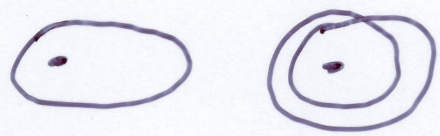
$$u = \frac{1}{r} \qquad du = -\frac{1}{r^2}\,dr \qquad u_0 = \frac{1}{r_0}$$

$$\frac{du}{dr} = -\frac{1}{r^2} = -u^2$$

$$V(r) = k\,r^{n+1} \longrightarrow k\,u^{-(n+1)}$$

$$\theta - \theta_0 = -\int_{u'=u_0}^{u} \frac{du'}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mk}{\ell^2}\,u'^{-(n+1)} - u'^{2}}}$$

Bertrand's theorem on closed orbits.

Closed orbits  | not closed 

Euler-Lagrange for r: $\quad m\ddot{r} = f(r) + \frac{\ell^2}{m r^3}$

Convert to u

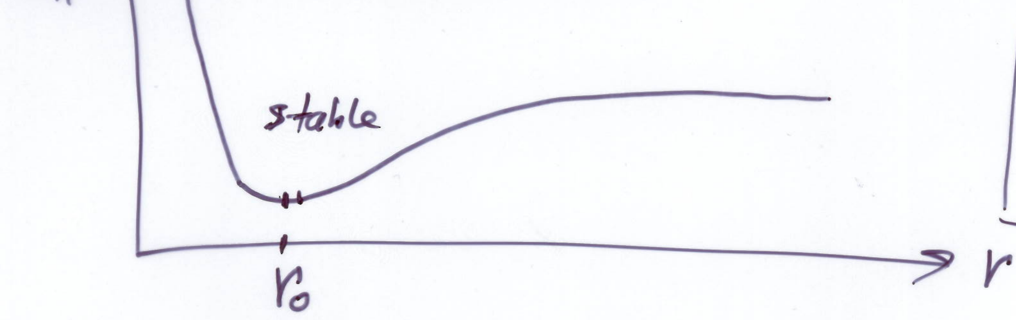
$$m\,\frac{\ell}{mr^2}\frac{d}{d\theta}\left[\frac{\ell}{mr^2}\frac{dr}{d\theta}\right] = -\frac{dV(r)}{dr} + \frac{\ell^2}{mr^3}$$

$$= \frac{\ell^2}{m}u^2\frac{d}{d\theta}\left[u^2 \underbrace{\frac{d}{d\theta}\left(\frac{1}{u}\right)}_{-\frac{1}{u^2}\frac{du}{d\theta}}\right] = -\frac{dV}{du}\frac{du}{dr} + \frac{\ell^2}{m}u^3$$

$$= -\frac{\ell^2}{m}u^2\frac{d^2u}{d\theta^2} = u^2\frac{dV\left(\frac{1}{u}\right)}{du} + \frac{\ell^2 u^3}{m}$$

$$\frac{d^2u}{d\theta^2} = -\frac{m}{\ell^2}\frac{dV\left(\frac{1}{u}\right)}{du} - u$$

Circular Orbits



If $r = r_0 \Rightarrow \dot{r} = 0,\ \ddot{r} = 0 \Rightarrow f(r_0) = -\dfrac{l^2}{m r_0^3}$

$\dot{r} = 0 \Rightarrow E = V(r_0) + \dfrac{l^2}{2 m r_0^2}$

$\frac{1}{2} m v_\theta^2$ (no $\frac{1}{2} m v_r^2$)

For stability, $V_{eff}(r)$ must be concave up

$$\frac{d^2 V_{eff}(r)}{dr^2} > 0 \Rightarrow -\frac{df}{dr}\bigg|_{r=r_0} + \frac{3l^2}{m r_0^4} > 0$$

$$\frac{df}{dr}\bigg|_{r=r_0} < -\frac{3 f(r_0)}{r_0}$$

If $f(r) = -k r^n$ this is $-k n r^{n-1} < 3 k r^{n-1}$

$$\Rightarrow \boxed{n > -3}$$

Simple harmonic motion for small deviations away from $r = r_0$ circular orbits

$u(\theta) = u_0 + a \cos(\beta\theta)$ small $a \ll u_0$

$\frac{1}{r_0}$ circles