Euler-Lagrange equation for r central force

$$① \quad m\ddot{r}(t) = f[r(t)] + \frac{\ell^2}{m[r(t)]^3}$$

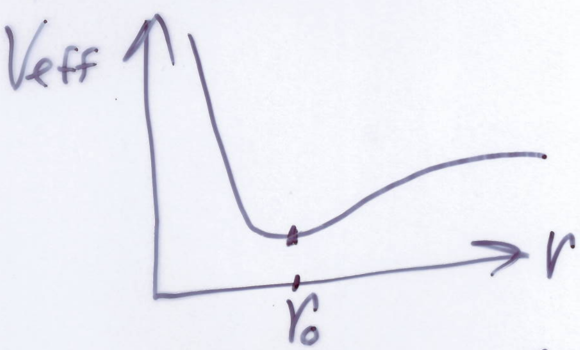
change variables $u \equiv \frac{1}{r}$, $du = -\frac{1}{r^2}dr$; $\frac{dr}{du} = -r^2 = -\frac{1}{u^2}$

$$② \quad \frac{d^2u(\theta)}{d\theta^2} + u(\theta) = -\frac{m}{\ell^2}\frac{dV(\frac{1}{u})}{du}$$

Aside $V(\frac{1}{u}) \neq V(u)$

e.g. $$\underset{\displaystyle W(u)}{\underset{\parallel}{V(r)}} = -\frac{k}{r} \neq V(u) = -\frac{k}{u} = -kr$$

Examine small oscillations about equilibrium



any function looks like a parabola near the minimum.

$\Longrightarrow$ Look for simple harmonic motion SHM

Define $u(\theta) = u_0 + a\cos(\beta\theta)$

small a, $a \ll u_0$; $u_0 \equiv \frac{1}{r_0}$

$$\frac{d^2u}{d\theta^2} = -\beta^2 a\cos(\beta\theta)$$

$$\frac{dV}{du} = \frac{dV(r)}{dr}\cdot\frac{dr}{du} = (-f(r))\left(-\frac{1}{u^2}\right) = \frac{f(r)}{u^2}$$

② $$-\beta^2 a\cos(\beta\theta) + u_0 + a\cos(\beta\theta) = -\frac{m}{l^2}\frac{1}{u^2}\left[f(r_0) + \left.\frac{df}{dr}\right|_{r_0}(r-r_0) + \dots\right]$$

At equilibrium $r = r_0$, $\dot{r} = 0$, $\ddot{r} = 0$.

$$0 = f(r_0) + \frac{l^2}{m r_0^3} \Rightarrow f(r_0) = -\frac{l^2 u_0^3}{m}$$

$$r - r_0 = \frac{1}{u} - \frac{1}{u_0} = \frac{1}{u_0 + a\cos(\beta\theta)} - \frac{1}{u_0} = \frac{1}{u_0\left[1 + \frac{a}{u_0}\cos(\beta\theta)\right]} - \frac{1}{u_0}$$

$$= \frac{1}{u_0}\left[1 - \frac{a}{u_0}\cos(\beta\theta)\right] - \frac{1}{u_0} = -\frac{a}{u_0^2}\cos(\beta\theta)$$

$$\frac{1}{u^2} = \frac{1}{(u_0 + a\cos(\beta\theta))^2} = \frac{1}{u_0^2}\frac{1}{\left[1 + \frac{a}{u_0}\cos(\beta\theta)\right]^2}$$

$$= \frac{1}{u_0^2}\left[1 - 2\frac{a}{u_0}\cos(\beta\theta)\right]$$

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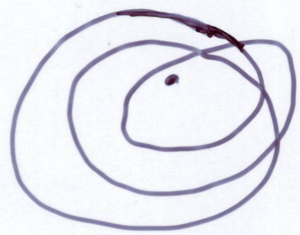
② $$-\beta^2 a\cos(\beta\theta) + u_0 + a\cos(\beta\theta) = -\frac{m}{l^2}\frac{1}{u_0^2}\left[1 - 2\frac{a}{u_0}\cos(\beta\theta)\right] \cdot$$

$$\cdot \left[-\frac{l^2 u_0^3}{m} + \left.\frac{df}{dr}\right|_{r_0}\left(-\frac{a}{u_0^2}\cos(\beta\theta)\right)\right]$$

$$= u_0 - 2a\cos(\beta\theta) + \frac{m}{l^2 u_0^4} a\cos(\beta\theta)\left.\frac{df}{dr}\right|_{r_0} + O(a^2)$$

$$\Rightarrow -\beta^2 + 3 = \frac{m}{l^2 u_0^4}\left.\frac{df}{dr}\right|_{r_0} = -\left.\left(\frac{r}{f}\frac{df}{dr}\right)\right|_{r_0}$$

As θ goes from 0 to 2π, u goes through β cycles of radial oscillation. If β is rational $\frac{p}{q}$ (reduced), the orbit will close after q revolutions, $\theta = 2\pi q$

e.g. If $\beta = \frac{5}{3} \Rightarrow$  closes after 3 orbits.

$$d(\ln f) = \frac{df}{f}$$

$$d(\ln r) = \frac{dr}{r}$$

$$\beta^2 - 3 = \left.\frac{d \ln f}{d \ln r}\right|_{r_0}$$

$$\Rightarrow d \ln f = (\beta^2 - 3)\, d \ln r \Rightarrow \ln f = (\beta^2 - 3) \ln r + A$$

$$\Rightarrow e^{\ln f} = f = e^{(\beta^2-3)\ln r + A} = r^{\beta^2 - 3} \cdot e^{A}$$

$$\Rightarrow f(r) = \frac{-k}{r^{3-\beta^2}}$$

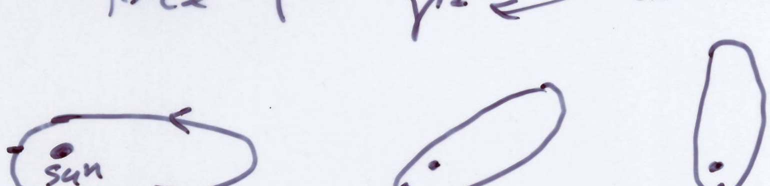
$\beta = 1$ gravity, $\beta = 2$ Hooke's Law spring

Bertrand's theorem

Any rational $\beta = \frac{p}{q}$ will give closed orbits to first order in $\underline{a}$. Only $\beta = 1, 2$ to second order in $\underline{a}$.

Mercury's perihelion precesses 5600 arcseconds per century. All but 43.1 $\frac{\text{arcsec}}{\text{cent}}$ are accounted for by other planets. General relativity gives the 43.1 $\frac{\text{arcsec}}{\text{cent.}}$

Force $f = \dfrac{-k}{r^2} \leftarrow$ 2.000 000 16 for Mercury.



Keplerian Orbits $f(r) = -\frac{k}{r^2}$ $V(r) = -\frac{k}{r}$

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mk}{\ell^2}u - u^2}}$$

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = -\frac{1}{\sqrt{-\gamma}} \arccos\left(\frac{-\beta - 2\gamma x}{\sqrt{\beta^2 - 4\alpha\gamma}}\right)$$

$$\alpha = \frac{2mE}{\ell^2} \qquad \beta = \frac{2mk}{\ell^2} \qquad \gamma = -1$$

$$\theta = \theta_0 - \arccos\left[\frac{\frac{\ell^2 u}{mk} - 1}{\sqrt{1 + \frac{2E\ell^2}{mk^2}}}\right]$$ solve for u

$$u = \frac{1}{r} = \frac{mk}{\ell^2}\left[1 + \sqrt{1 + \frac{2E\ell^2}{mk^2}}\cos(\theta - \theta_0)\right]$$

r_{min} at $\theta = \theta_0$ (perihelion)

r_{max} at $\theta = \theta_0 + \pi$ (aphelion)

$\uparrow$ turning points or apsides $\uparrow$

conic section with one focus at the origin

$$\frac{1}{r} = C\left[1 + \underset{\uparrow \text{ eccentricity}}{e} \cos(\theta - \theta_0)\right]$$

$$e = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$

$e > 1 \quad E > 0$ — hyperbola

$e = 1 \quad E = 0$ — parabola

$e < 1 \quad E < 0$ — ellipse

$e = 0 \quad E = -\frac{mk^2}{2\ell^2}$ — circle

Virial theorem $n = -2$: $\langle T \rangle = -\frac{1}{2}\langle V \rangle$

In a circular orbit, T and V are constants

$$T = -\frac{1}{2}V$$

$$E = T + V = -\frac{1}{2}V + V = \frac{1}{2}V = \frac{-k}{2r_0} = -\frac{mk^2}{2\ell^2}$$

$$f(r_0) = -\frac{\ell^2}{m r_0^3} = -\frac{k}{r_0^2}$$

Elliptical Orbits

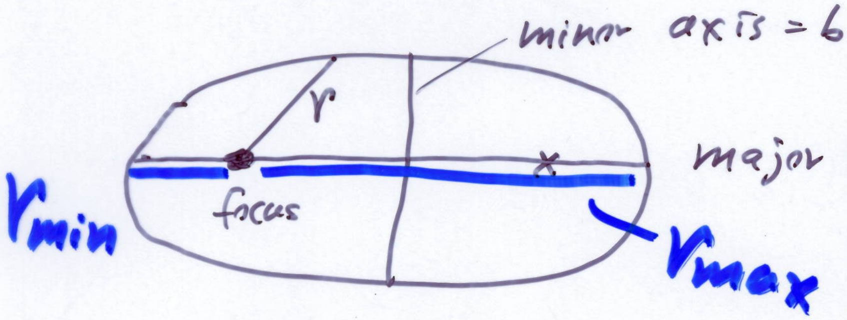
At apsides, r_{min} and r_{max}, $\dot{r} = 0$ instantaneously

$$E = T + V = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 + V$$

$$= \underbrace{\frac{1}{2}m\dot{r}^2}_{0} + \frac{\ell^2}{2mr^2} - \frac{k}{r}$$

$$\Rightarrow \frac{\ell^2}{2mr^2} - \frac{k}{r} - E = 0$$ quadratic equation in r

$$r_{\substack{max \\ min}} = \frac{-\frac{k}{E} \pm \sqrt{\frac{k^2}{E^2} + \frac{2\ell^2}{mE}}}{2}$$



$$\text{semi-major axis} = \frac{a}{2} = \frac{r_{min} + r_{max}}{2} = \frac{-k}{2E}$$

$$\text{eccentricity} \quad e = \sqrt{1 - \frac{\ell^2}{mka}}$$

$$\Rightarrow \quad \frac{\ell^2}{mk} = a(1-e^2)$$

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0)}$$ ← formula for ellipse
Kepler's first law