Elliptical Orbit Period

$$V(r) = -\frac{k}{r}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

$$t = \sqrt{\frac{m}{2k}} \int_{r_{min}}^{r} \frac{r'\,dr'}{\sqrt{r' - \frac{r'^2}{2a} - \frac{a(1-e^2)}{2}}}$$

change variable $r' = a(1 - e\cos\psi')$

ψ = eccentric anomaly

$\psi = 0$ at perihelion $= r_{min}$

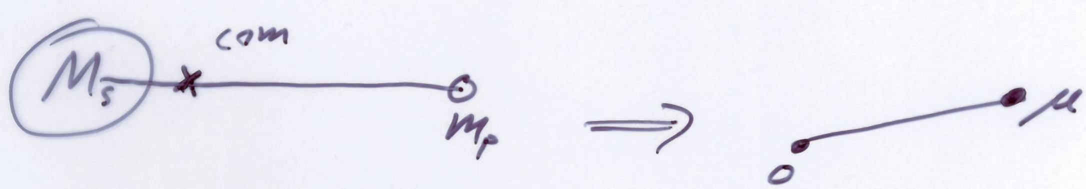
$\psi = \pi$ at aphelion $= r_{max}$

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} (1 - e\cos\psi')\,d\psi'$$

Integrate ψ from 0 to 2π.

Period $\tau = 2\pi a^{3/2}\sqrt{\frac{m}{k}}$ ← reduced mass μ

$$\mu = \frac{m_s m_p}{m_s + m_p}$$



For planets orbiting the Sun $\quad f = -\frac{G M_s m_p}{r^2}$

$k = G M_s m_p$

s = sun
p = planet

Almost Kepler's Third Law

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(M_s + m_p)}}$$

To the extent that m_p can be neglected compared to the solar mass M_s,

$$\tau \approx \frac{2\pi a^{3/2}}{\sqrt{GM_s}} \propto a^{3/2} \leftarrow \text{Kepler's Third Law}$$

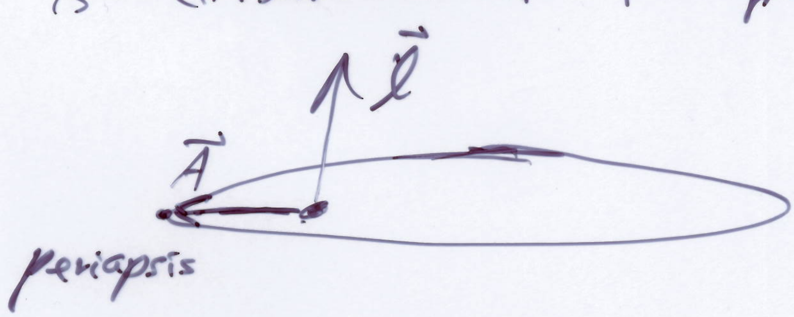
The solar system is the Sun

$$M_s = 99.86\% \text{ of } M_{\text{solar system}}.$$

$$M_{\text{Jupiter}} = 2.5\, M_{\text{all other planets together}}.$$

$$M_{\text{Jupiter}} = \frac{1}{1047} M_{\text{sun}}$$

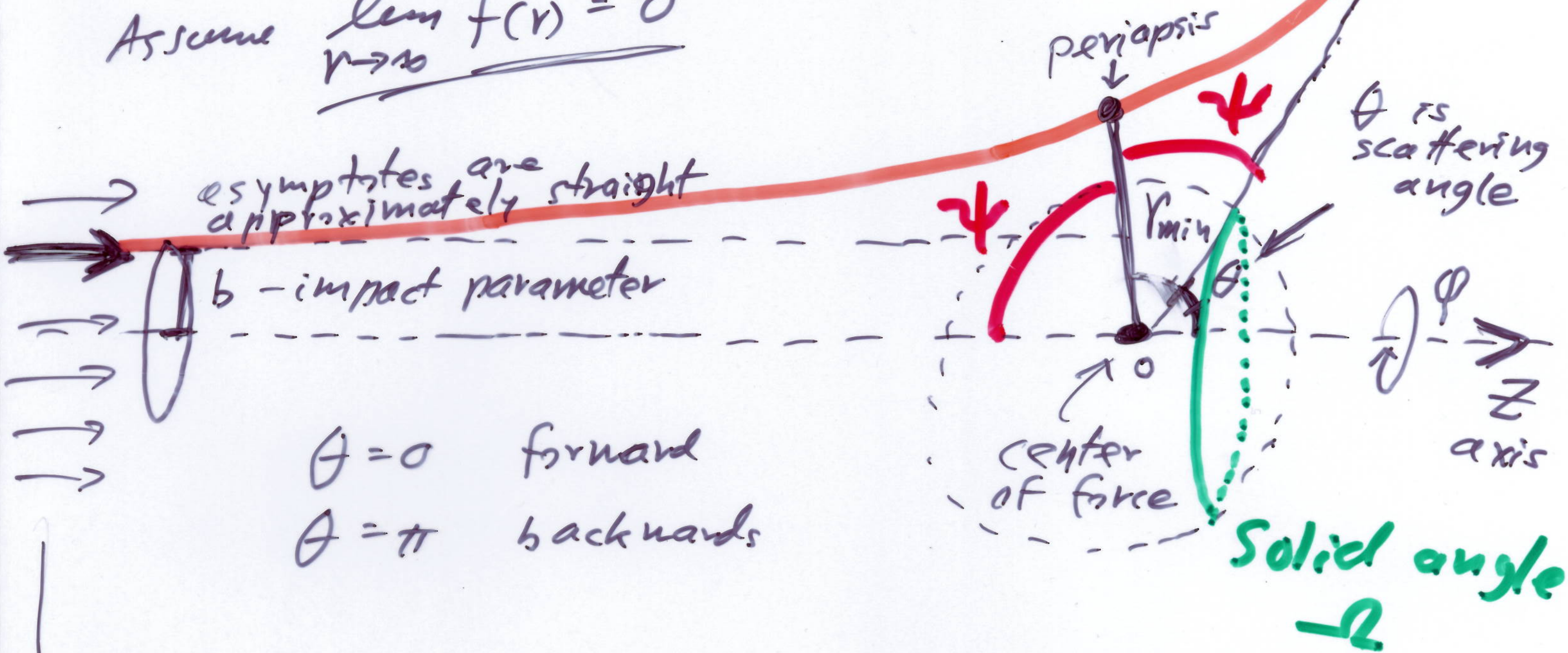
$\vec{A}$ = Laplace-Runge-Lenz vector is conserved if $f = -\frac{k}{r^2}$ exactly.



Scattering from a central force $V(r)$
attractive or repulsive

$$\vec{f}(r) = -\vec{\nabla} V(r)$$

Assume $\underline{\lim_{r \to \infty} f(r) = 0}$



$\theta = 0$ forward

$\theta = \pi$ backwards

I = intensity = luminosity = # particles per unit area (perpendicular to beam) per unit time

e.g. LHC design luminosity $= 10^{34}\ cm^{-2} s^{-1}$

Differential scattering cross section $\sigma(\Omega)$
also $\frac{d\sigma}{d\Omega}$ in some places

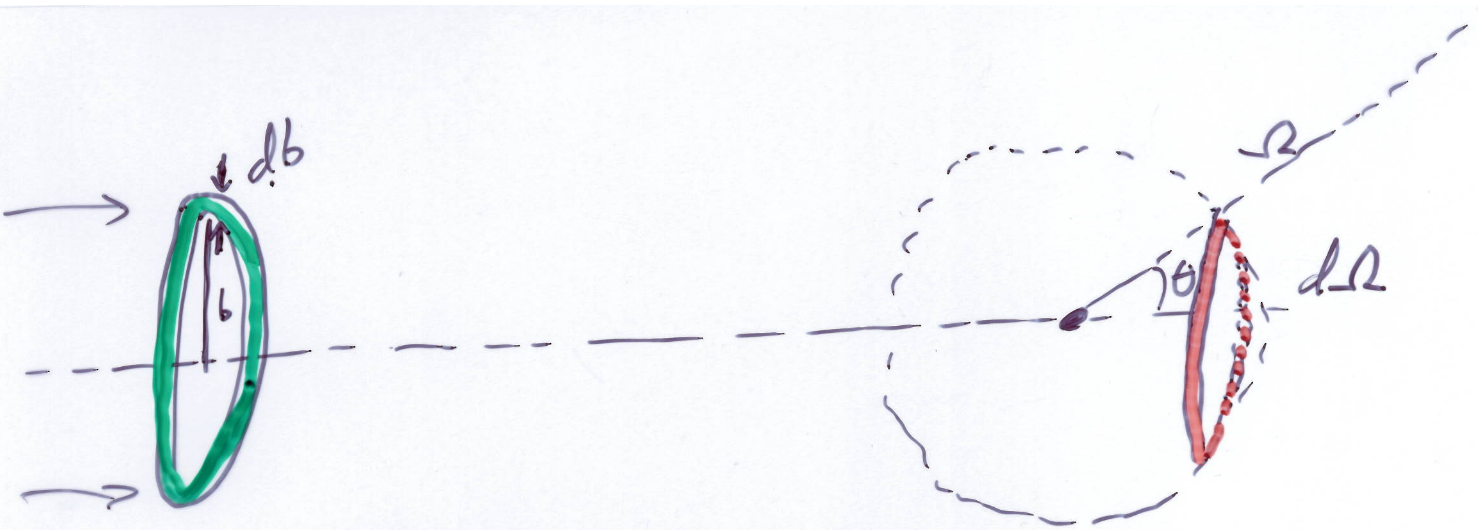
Total cross section $\sigma_{TOTAL} = \iint \sigma(\Omega)\, d\Omega$

$\sigma(\Omega)\, d\Omega$ = # particles scattered into $d\Omega$ around

$$\therefore \frac{\Omega \text{ per unit time}}{\text{incident intensity}}$$

$d\Omega = 2\pi \sin\theta\, d\theta$ because of azimuthal symmetry

$\sigma(\Omega)$ + σ_{TOTAL} have dimensions of area



incident speed of particles v_0

reduced mass m

Non-relativistic energy $E = T = \frac{m}{2} v_0^2$, $V = 0$

angular momentum $l = m v_0 b = b\sqrt{2mE}$

Assume different b's result in different θ's.

small $b \Rightarrow$ large θ

large $b \Rightarrow$ small θ

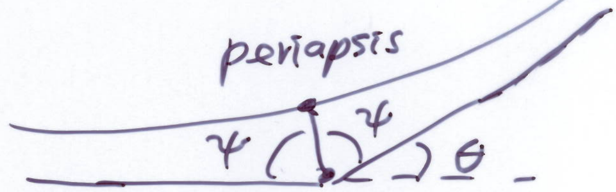
Particle number is conserved

# of incident particles with impact parameter between b and $b+db$ = # scattered particles with polar angle between θ and $\theta + d\theta$.

$$2\pi b |db| I = \sigma(\Omega) |d\Omega| I = 2\pi \sigma(\Omega) \sin\theta |d\theta| I$$

$$\sigma(\Omega) = \sigma(\theta,\varphi) = \sigma(\theta) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

$\theta = \pi - 2\psi$



scattering is symmetric about the periapsis.

Recall

$$\theta = \int_{r'=r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2mE}{l^2} - \frac{2mV(r')}{l^2} - \frac{1}{r'^2}}} + \theta_0$$

Incoming direction $\theta_0 = \pi$ and $r_0 = \infty$

$\theta = \pi - \psi$ at periapsis where $r = r_{min}$

$$\pi - \psi = \int_{r=\infty}^{r_{min}} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV(r)}{l^2} - \frac{1}{r^2}}} + \pi$$

$$\psi = \int_{r=r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV(r)}{l^2} - \frac{1}{r^2}}}$$

$l = b\sqrt{2mE}$

$$\theta(b) = \pi - 2\int_{r_{min}}^{\infty} \frac{b\,dr}{r\sqrt{r^2(1-\frac{V}{E}) - b^2}}$$

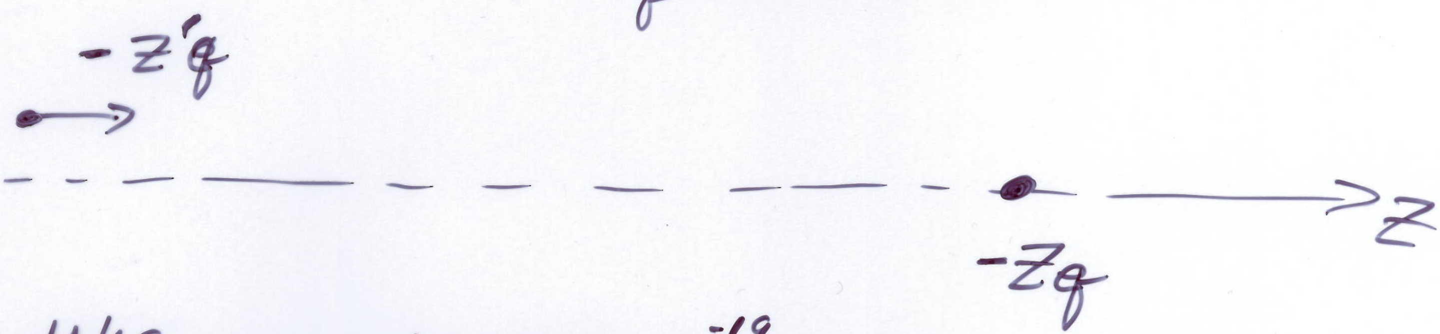
$u = \frac{1}{r}$

$$= \pi - 2\int_{0}^{u_{max}} \frac{b\,du}{\sqrt{1 - \frac{V(\frac{1}{u})}{E} - b^2u^2}}$$

Rutherford Scattering

Scattering in Coulomb electrostatic field
(ignoring radiation)

q = electron charge



MKS: $q = 1.60219 \times 10^{-19}$ coulombs

cgs: $q = 4.803 \times 10^{-10}$ e.s.u.

Coulomb force $\qquad \frac{1}{4\pi\epsilon_0} = \cancel{\frac{1}{4\pi\epsilon_0}} = 8.99 \times 10^{9}$ mks units.

MKS: $\quad f = \dfrac{ZZ'q^2}{4\pi\epsilon_0 r^2}$

cgs: $\quad f = \dfrac{ZZ'q^2}{r^2} = -\dfrac{k}{r^2} \Rightarrow k = -ZZ'q^2$

$E > 0 \Rightarrow$ hyperbolic orbit $\Rightarrow e > 1$

eccentricity

eccentricity $\; e = \sqrt{1 + \dfrac{2El^2}{mk^2}} = \sqrt{1 + \dfrac{2El^2}{m(ZZ'q^2)^2}}$

$$= \sqrt{1 + \left(\frac{2Eb}{ZZ'q^2}\right)^2}$$