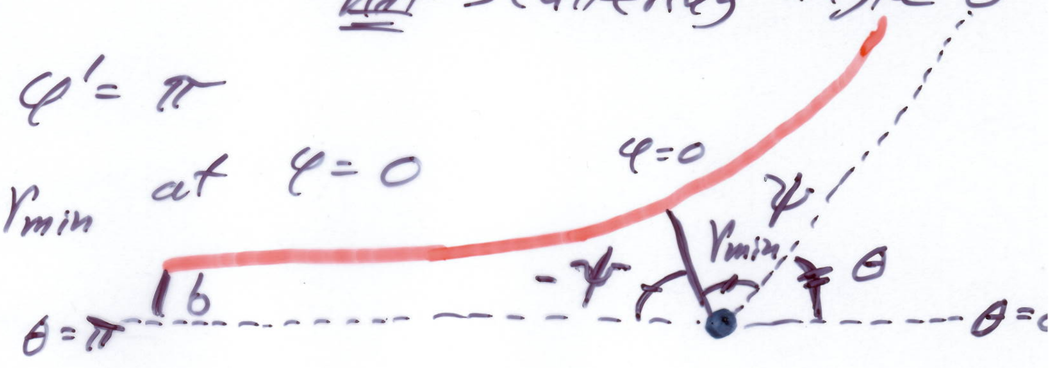
$$\frac{1}{r} = \frac{mk}{l^2}\left[1 + e\cos(\varphi - \varphi')\right]$$

$\uparrow$ <u>not</u> scattering angle θ

choose $\varphi' = \pi$

periapsis r_{min} at $\varphi = 0$



$$k = -ZZ'q^2 \text{ (cgs)} = \frac{-ZZ'q^2}{4\pi\epsilon_0} \text{ (mks)}$$

$$\frac{1}{r} = \frac{mZZ'q^2}{l^2}\left[e\cos\varphi - 1\right]$$

Get ψ this way

$\left.\begin{matrix}\text{incoming}\\ \text{outgoing}\end{matrix}\right\}$ direction $= \varphi = \pm\psi$ at $r \to \infty$

$$e\cos(\pm\psi) - 1 = 0 \Rightarrow \psi = \pm\arccos\left(\frac{1}{e}\right)$$

$$2\psi + \theta = \pi \Longrightarrow \theta = \pi - 2\psi$$

$$\psi = \frac{\pi}{2} - \frac{\theta}{2} \Longrightarrow \cos\psi = \sin\left(\frac{\theta}{2}\right)$$

$$\sin\left(\frac{\theta}{2}\right) = \cos\psi = \frac{1}{e}$$

Consider

$$\cot^2\left(\frac{\theta}{2}\right) = \frac{\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} = \frac{1-\sin^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} = \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} - 1$$

$$= e^2 - 1$$

$$\cot\left(\frac{\theta}{2}\right) = \sqrt{e^2-1} = \frac{2Eb}{ZZ'q^2}$$

$$b(\theta) = \frac{ZZ'q^2}{2E}\cot\left(\frac{\theta}{2}\right)$$

Remember Differential Scattering Cross Section

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\frac{db}{d\theta} = \frac{-ZZ'q^2}{2E}\csc^2\left(\frac{\theta}{2}\right)\frac{1}{2}$$

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\sigma(\theta) = \frac{\frac{ZZ'q^2}{2E}\cot\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\,\frac{ZZ'q^2}{2E}\csc^2\left(\frac{\theta}{2}\right)\frac{1}{2}$$

$$\sigma(\theta) = \frac{1}{4}\left(\frac{ZZ'q^2}{2E}\right)^2\csc^4\left(\frac{\theta}{2}\right)$$

Rutherford

Total cross section

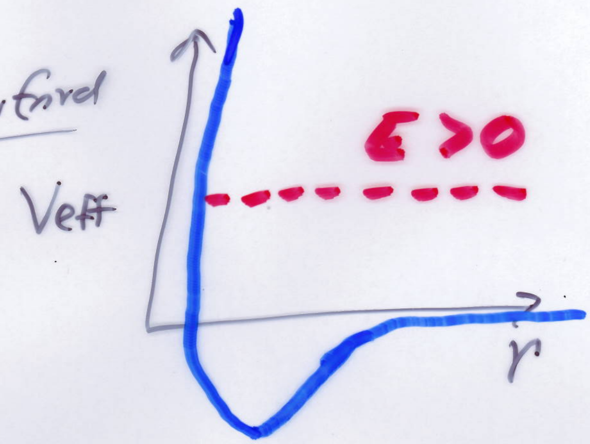
$$\sigma_{tot} = \int \sigma(\theta) \sin\theta \, d\theta = \infty$$

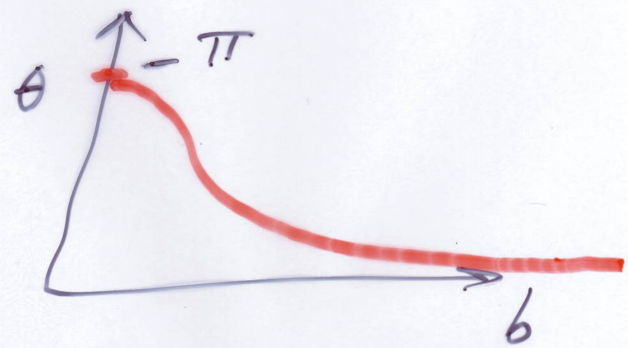
Coulomb force $\sim \frac{1}{r^2}$ out to infinity

__all__ incoming particles are deflected away from $\theta = 0$ (forward direction)

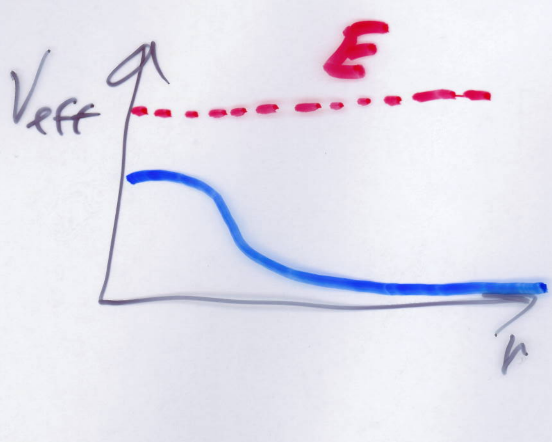
What if more than one $\underline{b}$ results in the same scattering angle θ?

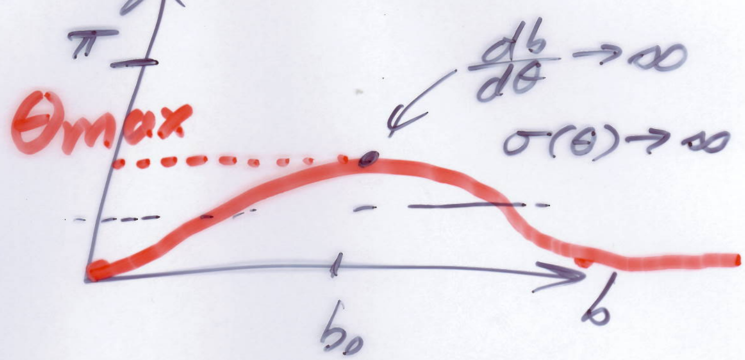
<u>Rutherford</u>





$$\left.\frac{d\theta}{db}\right|_{b_0} = 0$$

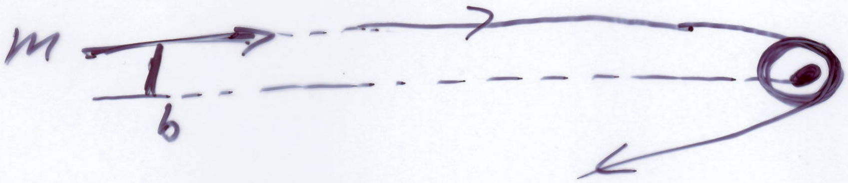




Rainbow scattering

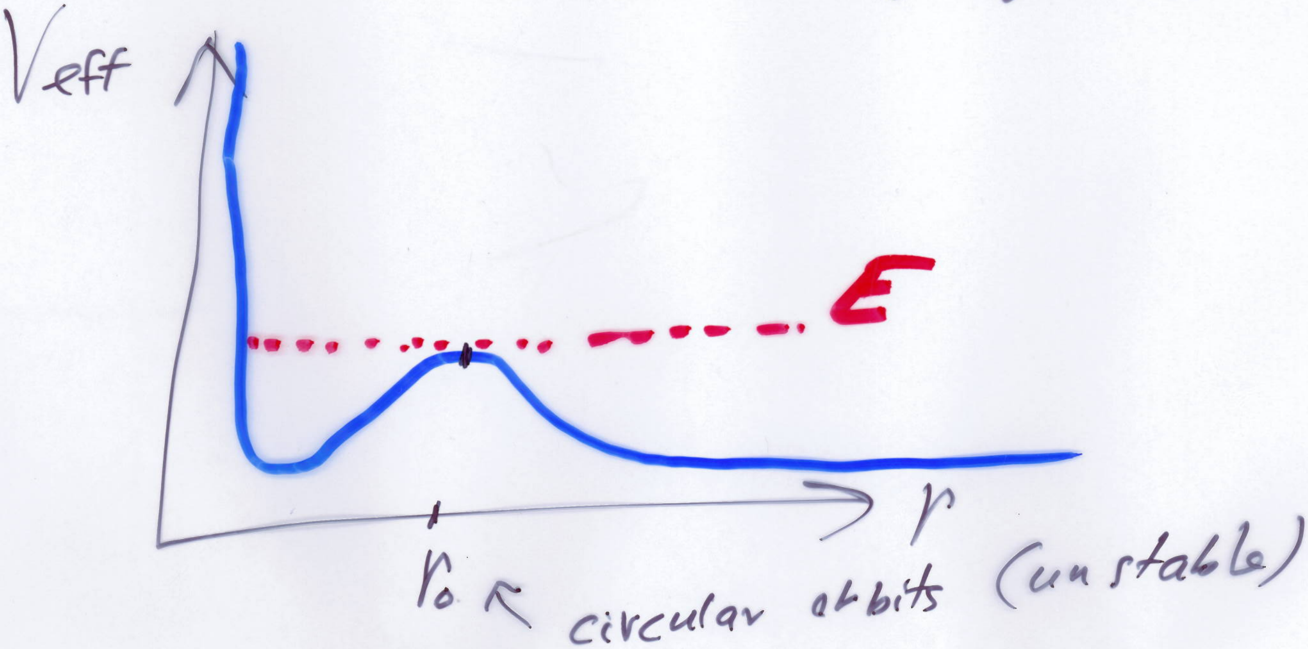
$$\sigma(\theta) = \sum_i \frac{b_i}{\sin\theta_i} \left| \frac{db_i}{d\theta_i} \right|$$

Attractive Scattering



① scattering angle
$\theta > \pi$

solution: subtract
$2\pi n$ so that
$0 \le \theta \le \pi$



Difference between the red curve and the blue curve is $\frac{m}{2}\dot{r}^2$

$\Rightarrow$ can make $\dot{r}$ arbitrarily small but $\dot{\theta} \neq 0$.

Exploring Black Holes

Introduction to General Relativity

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Massachusetts Institute of Technology

John Archibald Wheeler

Princeton University

Addison Wesley Longman

An imprint of Addison Wesley Longman

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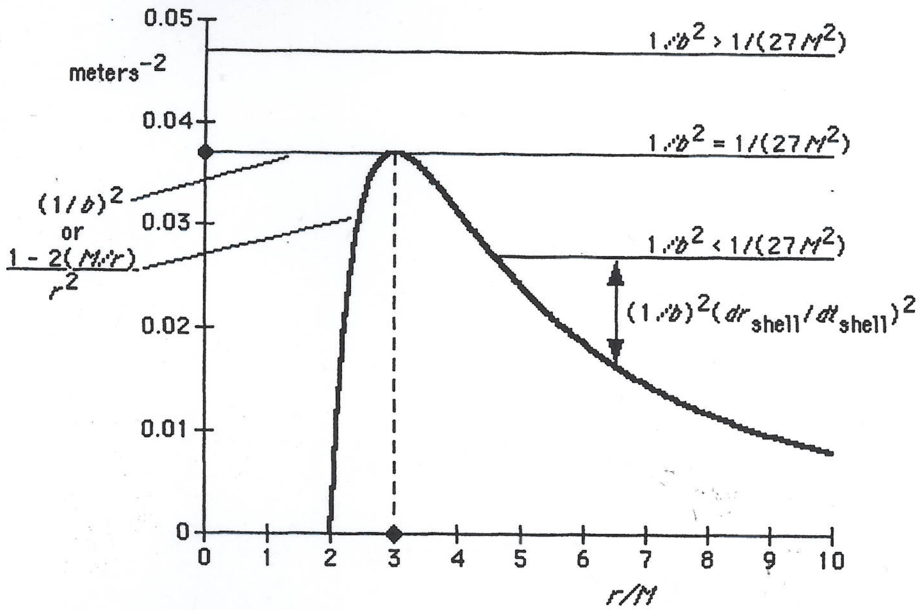

Figure 5 *Computer output: Square of the effective potential for light near a black hole. The same effective potential holds for light of all frequencies. There is no minimum in this potential, therefore no stable circular orbit for light. Trajectories can be described using the horizontal lines corresponding to different values of the quantity $1/b^2$. For a small value of the impact parameter b (large enough value of 1/b: top horizontal line), the light enters the black hole. For a large value of the impact parameter b (small enough value of 1/b: bottom horizontal line), the in-falling light reverses its radial component of motion and escapes to infinity. For the critical impact parameter ($b_{critical} = 27^{1/2}M = 5.20M$, line grazing the top of the effective potential) the light enters a circular knife-edge orbit of radius r = 3M and may orbit the black hole for part of an orbit or for many turns before it escapes or plunges. Figure 6 shows schematically these three orbits themselves. (Note: Here we plot the square of the effective potential, whereas in the figures in Chapter 4 for particles with mass we plotted effective potential itself.)*

7 Schwarzschild Maps of the Motion of Light

The larger view that no observer observes!

A light flash moving under the influence of a spherically symmetric center of attraction of given mass *M* has an orbit whose size and shape, praise be, depends on only a single quantity, the impact parameter *b*.

Size and shape of orbit depends only on *b*.

The trajectory of a light flash near a black hole lends itself to a simple description using the effective potential. For example, Figure 6 is what we call a **Schwarzschild map** of the orbits of light for three sample values of the impact parameter *b*. The Schwarzschild map shows three light trajectories as a function of Schwarzschild bookkeeper coordinates *r* and ϕ. Figure 5 traces the radial motions along three such trajectories using the effective potential.

Describe orbits with "Schwarzschild map."

From these figures we can derive a qualitative description of the trajectory of any light pulse, no matter what the value of its impact parameter *b*. The more formal—and accurate—Schwarzschild map of the trajectory comes from integrating equations [14] and [15] themselves.

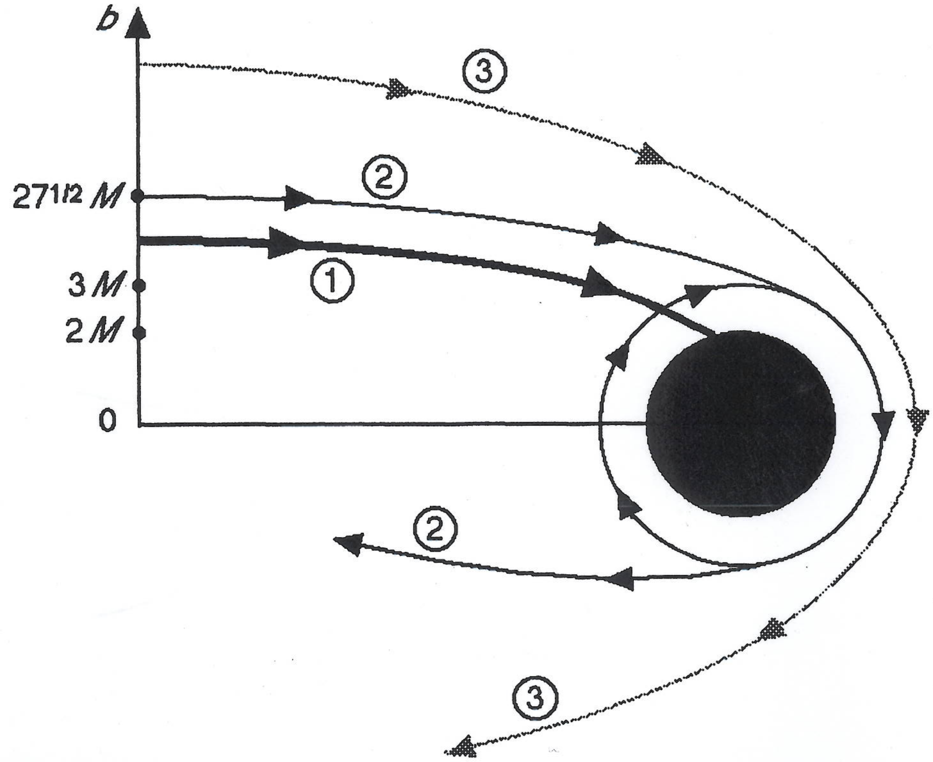

Figure 6 ***Schematic*** *Schwarzschild map of light trajectories around a black hole for the three values of the impact parameter b shown in Figure 5, page 5-13.*

1. Light is captured for b less than the critical value $27^{1/2}$ *M.*

2. For critical impact parameter, light teeters on unstable circular orbit at r = 3M. Eventually the light will plunge into the black hole—or escape to infinity, as shown.

3. For larger impact parameters, the trajectory is deflected but light is not captured.

SAMPLE PROBLEM 2 Escaping Light Flash?

Does the laser pulse described in Sample Problem 1, page 5-10, escape the black hole?

SOLUTION

That pulse had an impact parameter $b = 5.59\ M$. This value is greater than the critical impact parameter $b_{critical} = (27)^{1/2}\ M = 5.20\ M$. So that pulse escapes from the black hole.

The simplicity and power of the effective potential is witnessed by the brevity of this sample problem, which is the shortest in the book!

Schwarzschild map does not predict what shell observer sees.

Figures 5 and 6 do *not* tell us what we would see if we stood on a spherical shell near a black hole nor what color light we would perceive. Those figures focus on a Schwarzschild map, a plot artificially constructed from the accounting entries of the Schwarzschild bookkeeper, using coordinates *r*, ϕ, and *t*. The shell observer does not agree with the Schwarzschild bookkeeper about the direction of motion of these light beams. He does not