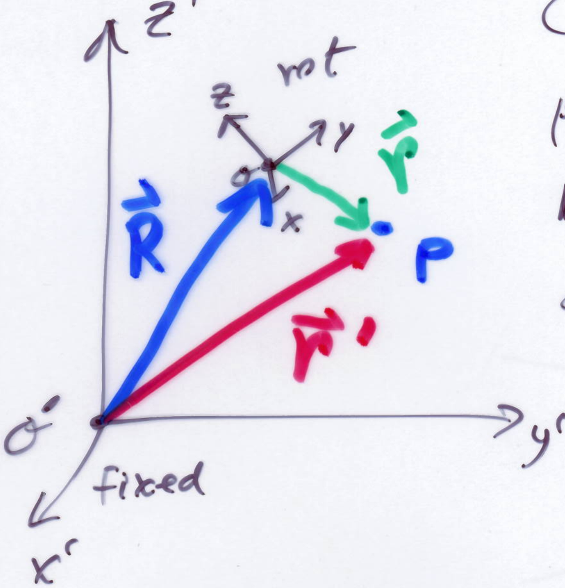
Non-inertial Reference Frames



Consider two coordinate frames: primed frame is inertial, where Newton's 1st and 2nd laws hold; and the unprimed is possibly non-inertial and labeled "rot" for rotating.

O could have some velocity with respect to O'. If this velocity $\left(\frac{d\vec{R}}{dt}\right)_{fixed}$ is constant then O is also inertial.

O could be rotating with angular velocity $\vec{\omega}$ as seen by an observer in O'

O could by accelerating away from O'

The point P can be at rest in O or P can be moving in frame O. The movement will look different in O'

1) Suppose P is at rest in $\mathcal{O}$

$$\vec{v}_{rot} \equiv \left(\frac{d\vec{r}}{dt}\right)_{rot} = 0$$

What is $\vec{v}_{fixed} \equiv \left(\dfrac{d\vec{r}'}{dt}\right)_{fixed}$?

It depends on the motion of $\mathcal{O}$ with respect to $\mathcal{O}'$

1a) $\mathcal{O}$ has velocity $\left(\dfrac{d\vec{R}}{dt}\right)_{fixed}$ w.r.t $\mathcal{O}'$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} \leftarrow \text{frame velocity } \vec{V}$$

1b) $\mathcal{O}$ is not translating, but it is rotating w.r.t. $\mathcal{O}'$ with angular velocity $\vec{\omega}$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \vec{\omega}\times\vec{r}$$

1c) $\mathcal{O}$ is translating and rotating

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \vec{\omega}\times\vec{r}$$

2) P is not at rest in O: $\left(\frac{d\vec{r}}{dt}\right)_{rot} \neq 0$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \vec{\omega} \times \vec{r}$$

$$\vec{v}'_{fixed} \equiv \vec{v}_{rot} + \vec{V} + \vec{\omega} \times \vec{r}$$

$$\vec{r}' = \vec{R} + \vec{r}$$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{fixed}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}$$

This will hold for any vector $\vec{Q}$

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q}}$$

e.g. Angular Acceleration

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fixed} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \underbrace{\vec{\omega}\times\vec{\omega}}_{zero}$$

$$\vec{\alpha}' = \vec{\alpha} \quad \text{or} \quad \dot{\vec{\omega}}' = \dot{\vec{\omega}}$$

Newton's 2nd Law

$$\sum \vec{F}_{real} = m\,\vec{a}'_{fixed} = m\left[\frac{d}{dt}\left(\frac{d\vec{r}'}{dt}\right)_{fixed}\right]_{fixed}$$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \vec{\omega}\times\vec{r}$$

$$\underset{\vec{a}'_{fixed}}{\overset{\|}{\left[\frac{d}{dt}\left(\frac{d\vec{r}'}{dt}\right)_{fixed}\right]_{fixed}}} = \underline{\left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{rot}\right]_{fixed}} + \underline{\left[\frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right)_{fixed}\right]_{fixed}}$$

$$+ \underline{\left[\frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right)\right]_{fixed}}$$

Goal: only fixed on left, only rot on right.

First term on right

$$\left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{rot}\right]_{fixed} = \underset{\vec{a}_{rot}}{\underset{\|}{\left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{rot}\right]_{rot}}} + \underset{\vec{\omega}\times\vec{v}_{rot}}{\underset{\|}{\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot}}}$$

$$\left[\frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right)_{fixed}\right]_{fixed} = \ddot{\vec{R}}_{fixed} = \vec{A}_{fixed}$$

is the frame acceleration.

$$\left[\frac{d}{dt}(\vec{\omega}\times\vec{r})\right]_{fixed} = \dot{\vec{\omega}}\times\vec{r} + \underline{\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{fixed}}$$

$\uparrow$ same in both frames

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}$$

$$= \vec{v}_{rot} + \vec{\omega}\times\vec{r}$$

$$\left[\frac{d}{dt}(\vec{\omega}\times\vec{r})\right]_{fixed} = \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{v}_{rot} + \vec{\omega}\times\vec{r})$$

$$= \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{v}_{rot} + \vec{\omega}\times\vec{r})$$

All together

$$\vec{a}'_{fixed} = \vec{a}_{rot} + \vec{\omega} \times \vec{v}_{rot} + \vec{A}_{fixed} + \dot{\vec{\omega}} \times \vec{r}$$
$$+ \vec{\omega} \times \vec{v}_{rot} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$\vec{r}$, $\vec{v}_{rot}$, $\vec{a}_{rot}$ all measured by a non-inertial observer in $\mathcal{O}$

What happens when a non-inertial observer tries to apply Newton's 2^{nd} Law.

$$m\,\vec{a}_{rot} = \sum \vec{F}_{effective}$$

$$= \underbrace{m\vec{a}'_{fixed}}_{\sum \vec{F}_{real}} \underbrace{- m\vec{A}_{fixed} - 2m\vec{\omega} \times \vec{v}_{rot} - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{"fictitious" forces}}$$

$-m\vec{A}_{fixed}$: "translational force" : e.g. you are thrown back in your seat when you step on the gas.

$-m\dot{\vec{\omega}}\times\vec{r}$: "azimuthal force" : results from angular acceleration

---

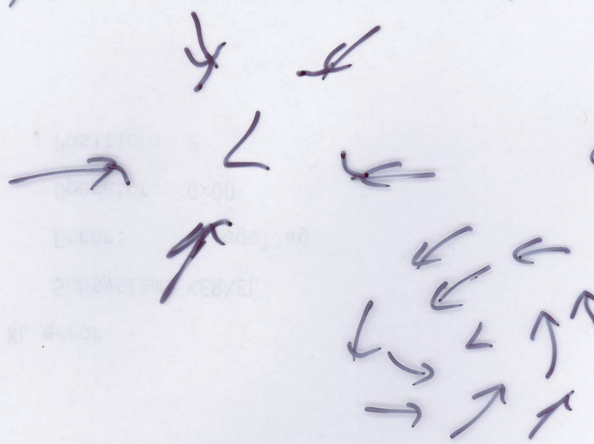
$-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$: centrifugal force : e.g. you are thrown to the outside of a curve when turn a corner. <u>or</u> rinse cycle of washer.

$-2m\vec{\omega}\times\vec{v}_{rot}$: Coriolis force : depends on velocity w.r.t. $\mathcal{O}$

e.g. try to walk on a merry-go-round

e.g. hurricanes

Northern Hemisphere

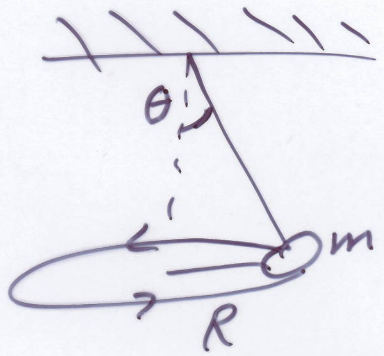


Coriolis force is to the right

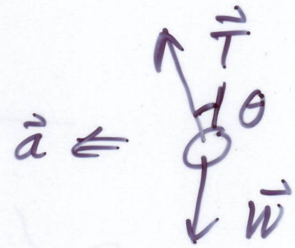
Counter ~~clock~~ clockwise rotation as seen from above.

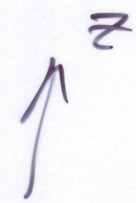
E.g. The Conical Pendulum

Inertial frame analysis



Free-body diagram





Cylindrical Polar Coordinates

$$\sum F_z = m\,\vec{a}_{z_{fixed}} = 0$$

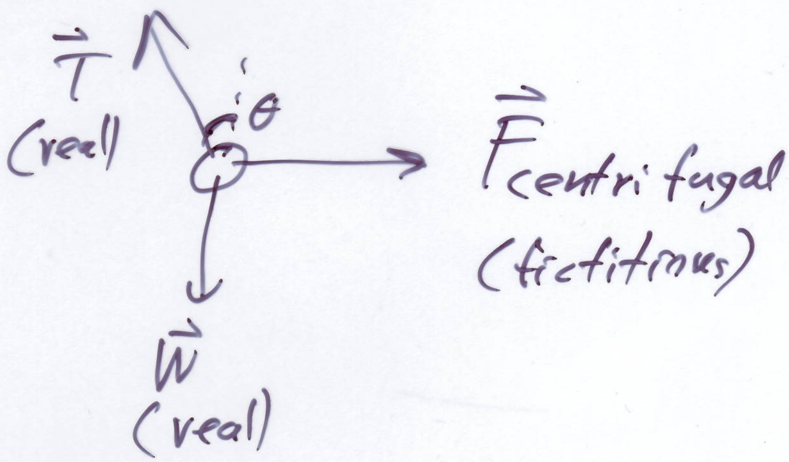
$$T\cos\theta - W = 0 \implies T = \frac{mg}{\cos\theta}$$

$$\sum F_\rho = m\,a_{\rho_{fixed}} \leftarrow \text{centripetal acceleration}$$

$$T\sin\theta = \frac{mv^2}{R}$$

$$\theta = \arctan\left(\frac{v^2}{Rg}\right)$$

Non-inertial (rotating) Frame Analysis



$$\sum F_z = m \cancel{a_{z_{rot}}}^{\to 0} = 0$$

$$T\cos\theta - W = 0$$

$$\sum F_\rho = m \cancel{a_{\rho_{rot}}}^{\to 0} = 0$$

$$-T\sin\theta + F_{centrifugal} = 0$$

$$\Updownarrow$$

$$-T\sin\theta + \frac{mv^2}{R} = 0$$

$$\theta = \arctan\left(\frac{v^2}{Rg}\right)$$

$$\vec{F}_{centrifugal} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$|\vec{\omega}\times\vec{r}| = \omega r\sin\theta = \omega R = v$$

$$|\vec{\omega}\times(\vec{\omega}\times\vec{r})| = \omega^2 R = \frac{v^2}{R}$$