An orthogonal rotation matrix $\underline{\underline{\lambda}}$ ($\underline{\underline{\lambda}}^T = \underline{\underline{\lambda}}^{-1}$) has at least one eigenvalue $= 1$

If rotation is proper (infinitesimally far from the identity) $\Rightarrow \det(\underline{\underline{\lambda}}) = +1$

$$\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{\mathbb{I}}} \Longrightarrow [\det(\underline{\underline{\lambda}})]^2 = 1$$

$$\Longrightarrow \det(\underline{\underline{\lambda}}) = \pm 1$$

$$(\underline{\underline{\lambda}} - \underline{\underline{\mathbb{I}}})\, \underline{\underline{\lambda}}^T = \underline{\underline{\lambda}}\,\underline{\underline{\lambda}}^T - \underline{\underline{\lambda}}^T = \underline{\underline{\mathbb{I}}} - \underline{\underline{\lambda}}^T$$

$$\det(\underline{\underline{\lambda}} - \underline{\underline{\mathbb{I}}}) \underbrace{\det(\underline{\underline{\lambda}}^T)}_{+1} = \det(\underline{\underline{\mathbb{I}}} - \underline{\underline{\lambda}}^T) = \det(\underline{\underline{\mathbb{I}}} - \underline{\underline{\lambda}})$$

$$\det(\underline{\underline{\lambda}} - \underline{\underline{\mathbb{I}}}) = (-1)^3 \det(\underline{\underline{\mathbb{I}}} - \underline{\underline{\lambda}})$$

$\leftarrow$ dimension of space

$$\Rightarrow \det(\underline{\underline{\lambda}} - \underline{\underline{\mathbb{I}}}) = 0$$

$$\underline{\underline{\lambda}}\, \vec{x} = 1\, \vec{x}$$ if $\vec{x}$ is an eigenvector

$\uparrow$ eigenvalue

$$= 1\, \underline{\underline{\mathbb{I}}}\, \vec{x}$$

non-trivial solutions $\vec{x} \neq \vec{0} \Rightarrow \underline{\underline{\lambda}} - \underline{\underline{\mathbb{I}}}$ can not be invertible.

$$q = 1a + \hat{i}b + \hat{j}c + \hat{k}d$$

$$\underline{\underline{\lambda}} = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

Composition

$$q = \cos\left(\frac{\alpha}{2}\right) + \hat{u}\sin\left(\frac{\alpha}{2}\right)$$

$$p = \cos\left(\frac{\beta}{2}\right) + \hat{w}\sin\left(\frac{\beta}{2}\right)$$

$$\vec{v}' = \quad p\, q\, \vec{v}\, q^* p^*$$

Pauli matrices

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_0 = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \qquad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Look like quaternions

$$1 \sim \sigma_0 \;, \quad \hat{i} \sim -i\sigma_1 \;, \quad \hat{j} = -i\sigma_3 \;, \quad \hat{k} = -i\sigma_3$$

Quaternions and Rotation

Complex numbers : $c = x + iy$ $\quad x, y \in \mathbb{R}$

$i = \sqrt{-1}$ imaginary unit

complex conjugation : $c^* = x - iy$

$|c|^2 = c \cdot c^*$ $\quad$ c^{-1} inverse : $c^{-1}c = 1$

$c^{-1} = c^*/(c \cdot c^*)$ unique

---

quaternions : $q = a + \hat{i}b + \hat{j}c + \hat{k}d$

$q^* = a - \hat{i}b - \hat{j}c - \hat{k}d$ $\quad a, b, c, d \in \mathbb{R}$

3 imaginary units

$\hat{i}^2 = -1 \qquad \hat{j}^2 = -1 \qquad \hat{k}^2 = -1$

$\hat{i}\hat{j} = \hat{k} \qquad \hat{j}\hat{k} = \hat{i} \qquad \hat{k}\hat{i} = \hat{j} \qquad \hat{i}\hat{k} = -\hat{j}$

properties : associative, not commutative.

$$q = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\left[\hat{i}u_x + \hat{j}u_y + \hat{k}u_z\right]$$

$\hat{u}$ is a unit vector $\hat{u} \cdot \hat{u} = 1$

q is a unit quaternion $q \cdot q^* = 1$

Rotate 3-vectors like this

$$\vec{v}' = q\,\vec{v}\,q^{-1} = q\,\vec{v}\,q^*$$

have

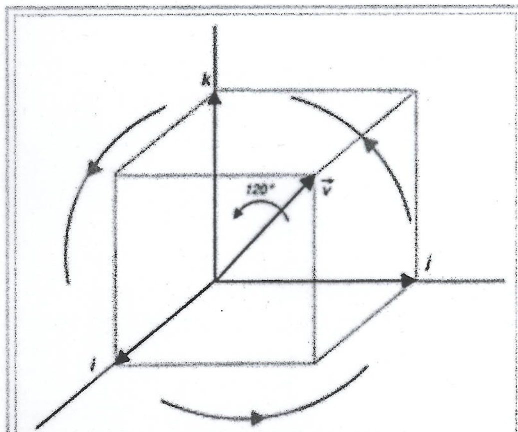
$$
\begin{aligned}
\vec{v}' &= \vec{v}\cos^2\tfrac{\alpha}{2} + (\vec{u}\vec{v} - \vec{v}\vec{u})\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2} - \vec{u}\vec{v}\vec{u}\sin^2\tfrac{\alpha}{2} \\
&= \vec{v}\cos^2\tfrac{\alpha}{2} + 2(\vec{u}\times\vec{v})\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2} - (\vec{v}(\vec{u}\cdot\vec{u}) - 2\vec{u}(\vec{u}\cdot\vec{v}))\sin^2\tfrac{\alpha}{2} \\
&= \vec{v}(\cos^2\tfrac{\alpha}{2} - \sin^2\tfrac{\alpha}{2}) + (\vec{u}\times\vec{v})(2\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}) + \vec{u}(\vec{u}\cdot\vec{v})(2\sin^2\tfrac{\alpha}{2}) \\
&= \vec{v}\cos\alpha + (\vec{u}\times\vec{v})\sin\alpha + \vec{u}(\vec{u}\cdot\vec{v})(1-\cos\alpha) \\
&= (\vec{v} - \vec{u}(\vec{u}\cdot\vec{v}))\cos\alpha + (\vec{u}\times\vec{v})\sin\alpha + \vec{u}(\vec{u}\cdot\vec{v}) \\
&= \vec{v}_{\perp}\cos\alpha + (\vec{u}\times\vec{v}_{\perp})\sin\alpha + \vec{v}_{\|}
\end{aligned}
$$

where $\vec{v}_{\perp}$ and $\vec{v}_{\|}$ are the components of $\vec{v}$ perpendicular and parallel to $\vec{u}$ respectively. This is the formula of a rotation by α around the $\vec{u}$ axis.

Example

The conjugation operation

Consider the rotation *f* around the axis $\vec{v} = \mathrm{i} + \mathrm{j} + \mathrm{k}$, with a rotation angle of 120°, or $^{2\pi}/_{3}$ radians.



A rotation of 120° around the first diagonal permutes i, j, and k cyclically.

$$\alpha = \frac{2\pi}{3}$$

The length of $\vec{v}$ is √3, the half angle is $^{\pi}/_{3}$ (60°) with cosine ½, (cos 60° = 0.5) and sine $^{\sqrt{3}}/_{2}$, (sin 60° ≈ 0.866). We are therefore dealing with a conjugation by the unit quaternion

$$
\begin{aligned}
u &= \cos\tfrac{\alpha}{2} + \sin\tfrac{\alpha}{2}\cdot\tfrac{1}{\|\vec{v}\|}\vec{v} \\
&= \cos\tfrac{\pi}{3} + \sin\tfrac{\pi}{3}\cdot\tfrac{1}{\sqrt{3}}\vec{v} \\
&= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{\sqrt{3}}\vec{v} \\
&= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\cdot\tfrac{\mathrm{i}+\mathrm{j}+\mathrm{k}}{\sqrt{3}} \\
&= \tfrac{1+\mathrm{i}+\mathrm{j}+\mathrm{k}}{2}
\end{aligned}
$$

If *f* is the rotation function,

$$f(a\mathrm{i} + b\mathrm{j} + c\mathrm{k}) = u(a\mathrm{i} + b\mathrm{j} + c\mathrm{k})u^{-1}$$

It can be proved that the inverse of a unit quaternion is obtained simply by changing the sign of its imaginary components. As a consequence,

$$u^{-1} = \frac{1 - \mathrm{i} - \mathrm{j} - \mathrm{k}}{2}$$

and

$$f(a\mathrm{i} + b\mathrm{j} + c\mathrm{k}) = \frac{1 + \mathrm{i} + \mathrm{j} + \mathrm{k}}{2}(a\mathrm{i} + b\mathrm{j} + c\mathrm{k})\frac{1 - \mathrm{i} - \mathrm{j} - \mathrm{k}}{2}$$

This can be simplified, using the ordinary rules for quaternion arithmetic, to

$$f(a\mathrm{i} + b\mathrm{j} + c\mathrm{k}) = c\mathrm{i} + a\mathrm{j} + b\mathrm{k}$$

As expected, the rotation corresponds to keeping a cube held fixed at one point, and rotating it 120° about the long diagonal through the fixed point (observe how the three axes are permuted cyclically).

Quaternion arithmetic in practice

Let's show how we reached the previous result. Let's develop the expression of f (in two stages), and apply the rules

$$\begin{aligned}
&\mathrm{ij} = \mathrm{k}, \quad \mathrm{ji} = -\mathrm{k}, \\
&\mathrm{jk} = \mathrm{i}, \quad \mathrm{kj} = -\mathrm{i}, \\
&\mathrm{ki} = \mathrm{j}, \quad \mathrm{ik} = -\mathrm{j}, \\
&\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1
\end{aligned}$$

It gives us:

Pauli matrices act in the fundamental representation on spinors — 2x1 objects $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$. Spinors describe spin-$\frac{1}{2}$ objects under rotation.

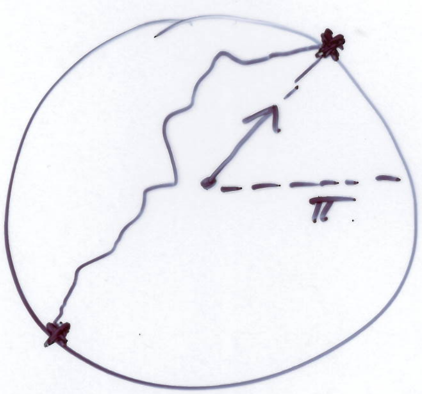
Spin $\frac{1}{2}$ Lie algebras : $su(2)$ is isomorphic $so(3)$.

$$\begin{array}{ccc} su(2) & & so(3) \\ \downarrow & & \downarrow \\ SU(2) & \neq & SO(3) \end{array}$$

Lie groups : $SU(2) \neq SO(3)$

$SU(2)$ is the double cover of $SO(3)$.

$SO(3)$

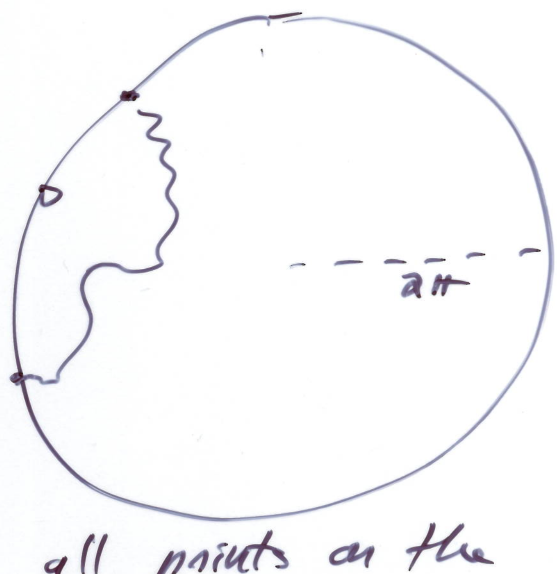


direction
magnitude = length

antipodal points are equivalent

doubly connected

$SU(2)$

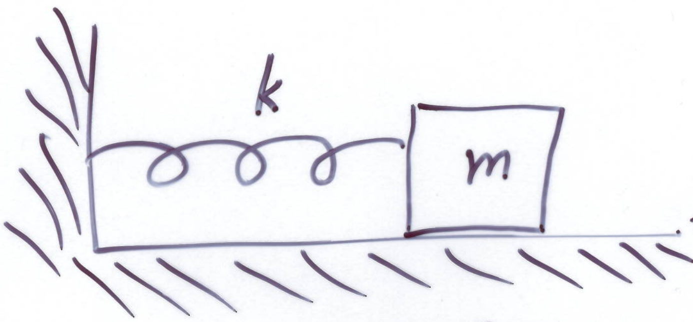


all points on the surface are equivalent

simply connected

Chapter 6: Oscillations

Consider a mass on a spring



Newton's 2^{nd} law $\quad \sum F_x = m a_x$

$$-kx = m\ddot{x}$$

x is a function of time t : $x(t)$

Standard form $\quad \ddot{x}(t) + \frac{k}{m} x(t) = 0$

2^{nd} order, linear (in x), ordinary, homogeneous differential equation.

e.g. Nonlinear $\quad \ddot{x}(t) + \frac{k}{m}[x(t)]^2 = 0$

or $\quad \sin[\ddot{x}(t)] + \frac{k}{m} x(t) = 0$

e.g. non homogeous

$$\ddot{x}(t) + \frac{k}{m} x(t) + 7 = 0$$

no x

Try an exponential solution

$$\left.\begin{aligned} x(t) &= A e^{rt} \\ \dot{x}(t) &= rA e^{rt} \\ \ddot{x}(t) &= r^2 A e^{rt} \end{aligned}\right\} \text{substitute}$$

$$\ddot{x}(t) + \frac{k}{m} x(t) = 0 \implies r^2 A e^{rt} + \frac{k}{m} A e^{rt} = 0$$

$e^{rt} \neq 0$ divide $\qquad A \neq 0$ nontrivial

characteristic equation: $r^2 + \frac{k}{m} = 0$

two roots: $r_{\pm} = \pm i \sqrt{\frac{k}{m}}$

Define $\omega_0 \equiv \sqrt{\frac{k}{m}}$ natural angular frequency of oscillation.

$$\begin{aligned} x(t) &= A_1 e^{+i\omega_0 t} + A_2 e^{-i\omega_0 t} \\ &= B_1 \sin(\omega_0 t) + B_2 \cos(\omega_0 t) \\ &= C_1 \sin(\omega_0 t + C_2) \\ &= D_1 \cos(\omega_0 t + D_2) \end{aligned}$$

2nd order $\Rightarrow$ 2 constants will fixed by initial conditions.