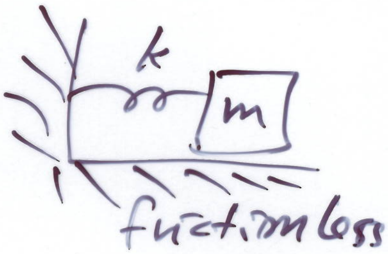
Undamped, undriven Simple Harmonic Oscillations

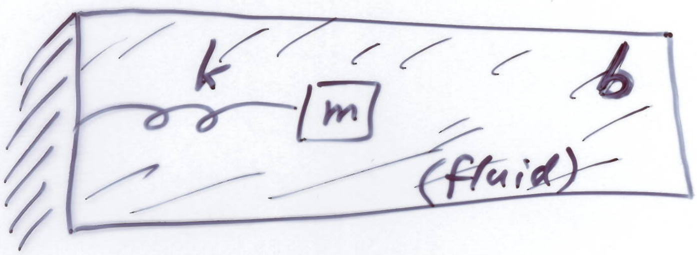


$$\ddot{x}(t) + \frac{k}{m} x(t) = 0$$

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t)$$

$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

Damped Oscillations



Aside: surface friction (kinetic)

$$\vec{f}_k = \mu_k N (-\hat{v})$$

viscous drag

$$f_{visc} = -b\vec{v} \checkmark$$

aerodynamic drag

$$f_{aero} = -c\, v^2 \hat{v}$$

Newton's 2^{nd} Law

$$\sum F = ma \qquad \omega_0^2 \equiv \frac{k}{m}$$

$$-kx - bv = ma \qquad \beta \equiv \frac{b}{2m}$$

$$-kx - b\dot{x} = m\ddot{x}$$

$$\Rightarrow \ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0$$

$$\Rightarrow \ddot{x}(t) + 2\beta\,\dot{x}(t) + \omega_0^2\,x(t) = 0$$

2^{nd} order, linear, homogeneous, ordinary differential equation.

Try ~~an~~ exponential solution

$$x(t) = Ae^{rt}$$

$$(r^2 + 2\beta r + \omega_0^2)\,Ae^{rt} = 0$$

$$\longrightarrow (r^2 + 2\beta r + \omega_0^2) = 0$$

$$r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = e^{-\beta t}\left[A_1 e^{+\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

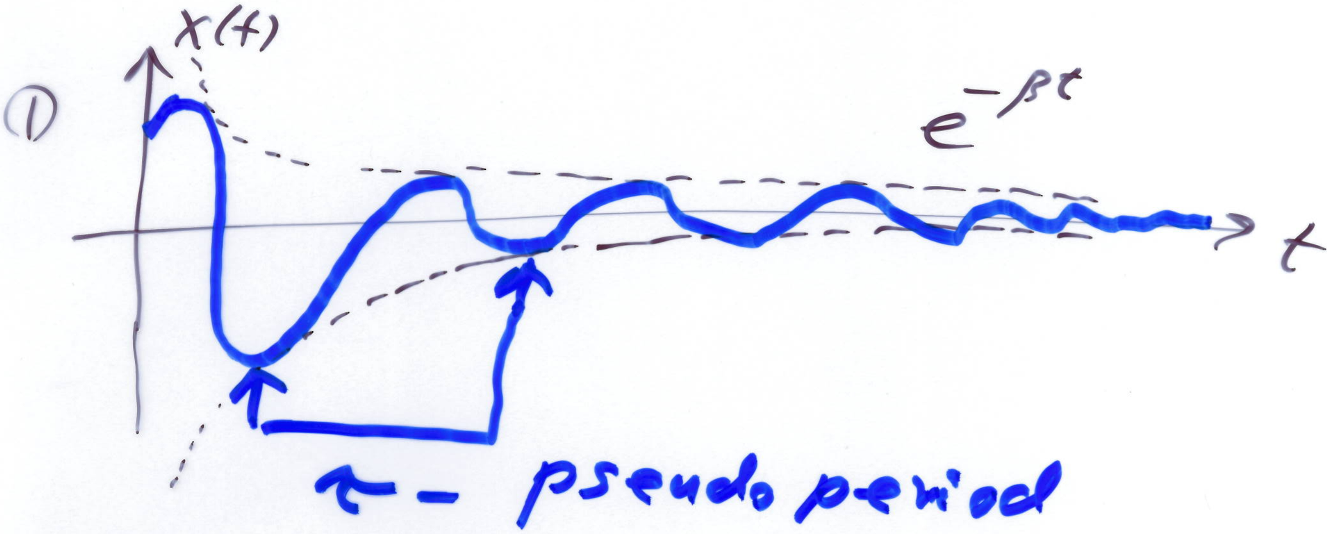
↑ damping envelope

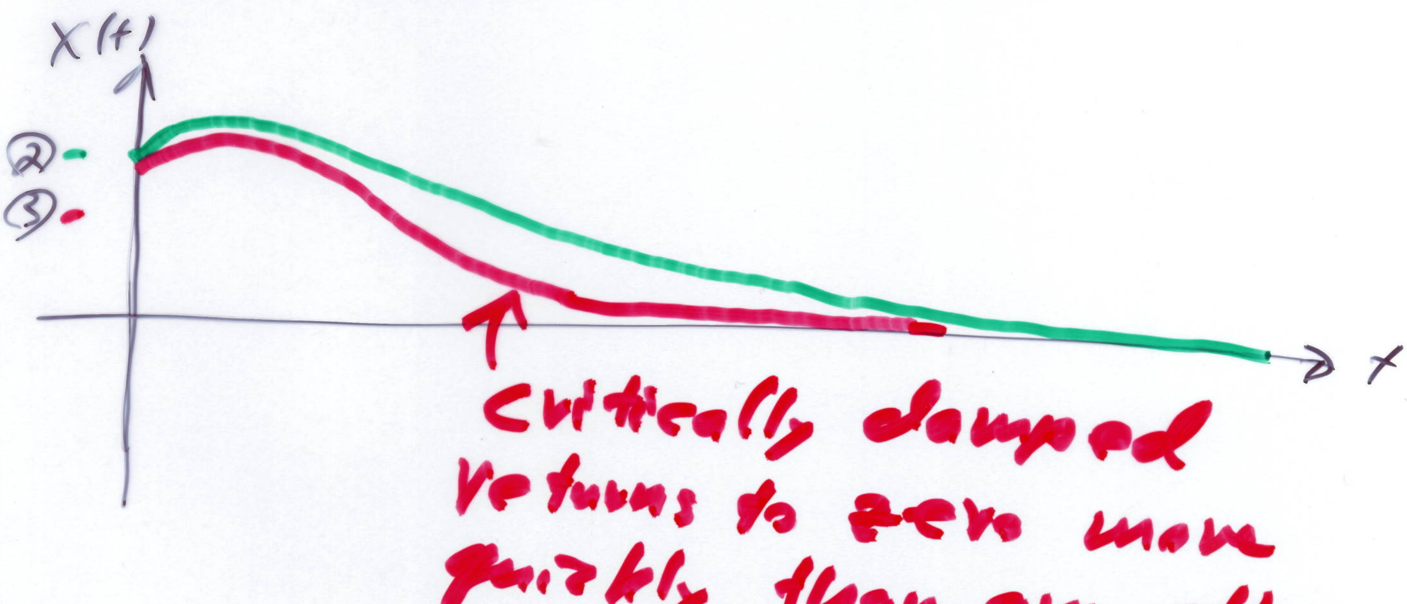
Three cases:

① $\omega_0 > \beta$ so $\sqrt{\beta^2-\omega_0^2}$ is imaginary → underdamped

② $\omega_0 < \beta$ so $\sqrt{\beta^2-\omega_0^2}$ is real → overdamped

③ $\omega_0 = \beta$ so $\sqrt{\beta^2-\omega_0^2}$ is zero → critically damped





① $\omega_0 > \beta$ underdamped

define $\omega_1^2 \equiv \omega_0^2 - \beta^2 > 0$

$\sqrt{\beta^2 - \omega_0^2} = \pm i \omega_1$ $\qquad \omega_1 < \omega_0$

$$x(t) = e^{-\beta t}\left[A_1 e^{+i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]$$

$$= e^{-\beta t}\left[B_1 \sin(\omega_1 t) + B_2 \cos(\omega_1 t)\right]$$

$$= e^{-\beta t}\left[C_1 \sin(\omega_1 t + C_2)\right]$$

or cos

every solution has two constants of integration which are fixed by initial conditions.

$\tau = \dfrac{2\pi}{\omega_1}$ = pseudoperiod

② $\omega_0 < \beta$ overdamped

Define $\omega_2^2 \equiv \beta^2 - \omega_0^2 > 0$

$$x(t) = e^{-\beta t}\left[A_1 e^{+\omega_2 t} + A_2 e^{-\omega_2 t}\right]$$

---

③ $\omega_0 = \beta$ critical damping

$$r_\pm \equiv -\beta \pm \sqrt{\beta^2 - \omega_0^2} \longrightarrow r_+ = -\beta = r_-$$

two <u>degenerate</u> roots

two solutions $A_1 e^{r_+ t}$ and $A_2 e^{r_- t}$

are no longer linearly independent.

Need a linearly independent solution

$\longrightarrow$ multiply by powers of t

Try $x(t) = A_1 e^{-\beta t} + A_2 t e^{-\beta t}$

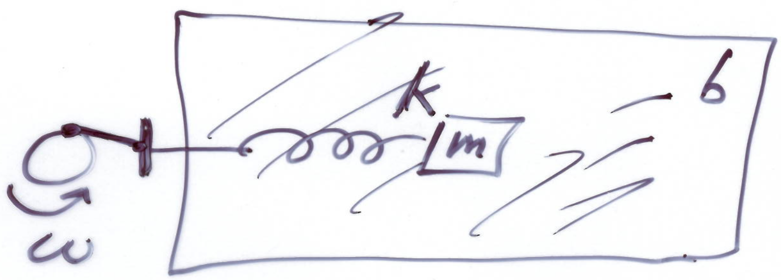
↑ check this

substitute into $\ddot{x}(t) + 2\beta \dot{x}(t) + \omega_0^2 x(t) = 0$

For β close to ω_0

$$e^{r_+ t} = e^{-\beta t + \epsilon t} = e^{-\beta t} \cdot e^{\epsilon t} = e^{-\beta t}\left[1 + \epsilon t + \dots\right]$$

Sinusoidally Driven Oscillations



$$A_0 = \frac{F_0}{m}$$

Newton's 2^{nd} Law

ang. driving frequency

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos(\omega t)$$

$$\Rightarrow \ddot{x}(t) + 2\beta \dot{x}(t) + \omega_0^2 x(t) = A_0 \cos(\omega t)$$

2^{nd} order, linear, non-homogeneous ordinary differential equation.

general solution = complementary solution + particular solution

$$X(t) = X_c(t) + X_p(t)$$

solution to the homogeneous equation

$$\ddot{x}_c(t) + 2\beta \dot{x}_c(t) + \omega_0^2 x_c(t) = 0$$

$$\frac{\text{transients } X_c(t)}{\text{steady state } x_p(t)}$$

$X_c(t)$ has two constants of integration A_1 + A_2 that are fixed by initial conditions. $X_p(t)$ is unique.

Define a differential operator

$$\mathcal{D} \equiv \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$$

$$\mathcal{D}[x_c(t)] = 0$$

$$\mathcal{D}[x_p(t)] = A_0 \cos(\omega t)$$

---

complementary solution

$$x_c(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

three cases....

---

particular solution - Guess

$$x_p(t) = D\cos(\omega t - \delta)$$

Response to forcing at the driving frequency ω. There is a possible phase shift δ.

D and δ are <u>not</u> arbitrary - they will be determined completely. Only A_1 and A_2 in the complementary solution are fixed by initial conditions.