Differential Operator

$$\mathcal{D} = \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) \cdots$$

$$\mathcal{D}[x_c(t)] = 0$$

$$\mathcal{D}[x_p(t)] = F_0 \cos(\omega t)$$

The set that gets mapped to zero by some transformation is called the "Kernel" or "nullspace."

e.g. $\underline{\underline{M}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\underline{\underline{M}} \cdot \vec{v} = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$ $\qquad$ Kernel is z-axis

Find a vector $\vec{v}$ of length 3 that maps onto $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow b = 2$, $a = -1$, use c to get $|\vec{v}| = 3$

$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$

Damped Driven Simple Harmonic Oscillator

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = \frac{F_0 \cos(\omega t)}{m}$$

general solution $x(t) = x_c(t) + x_p(t)$

$x_c(t)$ solves the homogeneous equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x_c(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

Guess $x_p(t) = D\cos(\omega t - \delta)$

$$\dot{x}_p(t) = -\omega D \sin(\omega t - \delta)$$

$$\ddot{x}_p(t) = -\omega^2 D \cos(\omega t - \delta)$$

$$D\left[-\omega^2 \underline{\cos(\omega t-\delta)} - 2\beta\omega\,\underline{\sin(\omega t-\delta)} + \omega_0^2\,\underline{\cos(\omega t-\delta)}\right] = \frac{F_0}{m}\cos(\omega t) \quad \text{for all time } t$$

$$\cos(\omega t)\cos\delta + \sin(\omega t)\sin\delta$$

$$\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta$$

$\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent

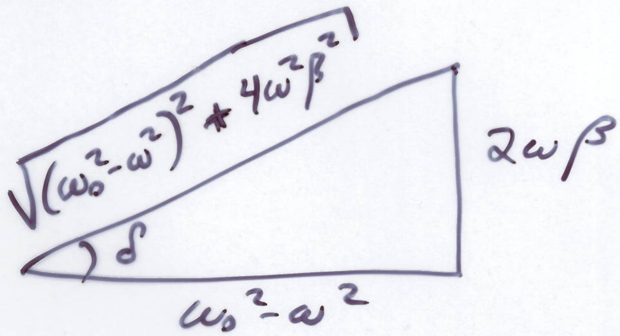
$$\left\{ \frac{F_0}{m} - D\left[(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta\right]\right\}\cos(\omega t)$$
$$- D\left[(\omega_0^2-\omega^2)\sin\delta - 2\omega\beta\cos\delta\right]\sin(\omega t) = 0$$

$$\Rightarrow \frac{F_0}{m} - D\left[(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta\right] = 0 \quad (1)$$

$$D\left[(\omega_0^2-\omega^2)\sin\delta - 2\omega\beta\cos\delta\right] = 0 \quad (2)$$

(2) $D \neq 0 \Rightarrow (\omega_0^2-\omega^2)\sin\delta = 2\omega\beta\cos\delta$

$$\tan\delta = \frac{2\omega\beta}{\omega_0^2-\omega^2} \Rightarrow \delta = \arctan\left[\frac{2\omega\beta}{\omega_0^2-\omega^2}\right]$$



$$\sin\delta = \frac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

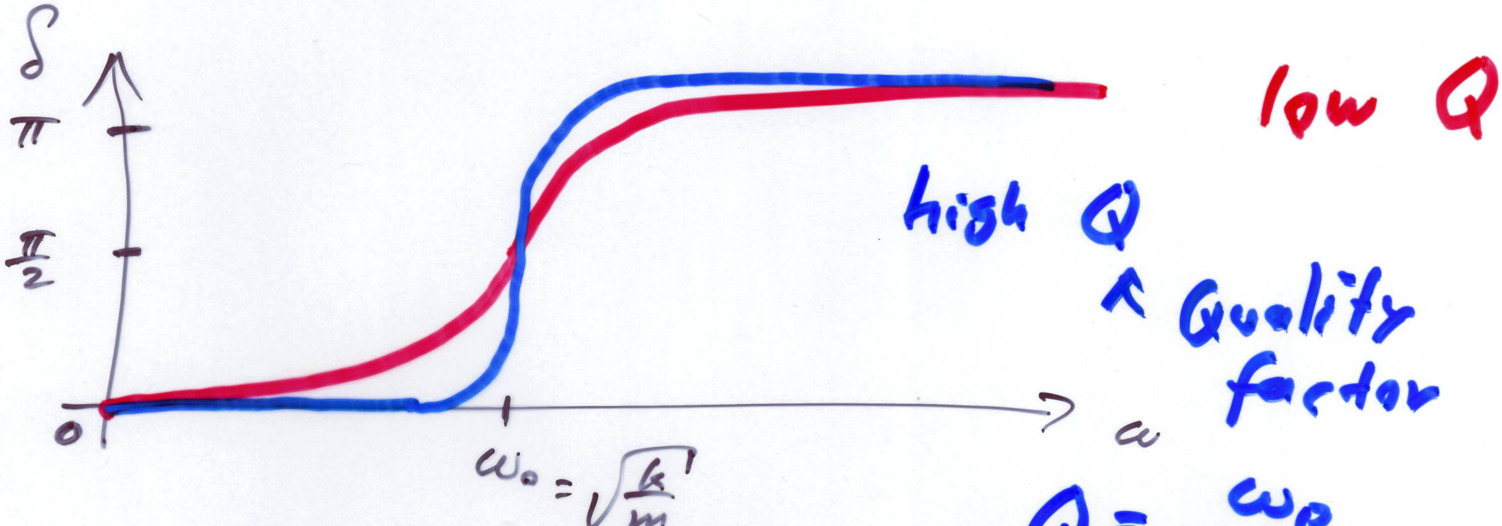
$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

substitute into equation (1) and solve for D.

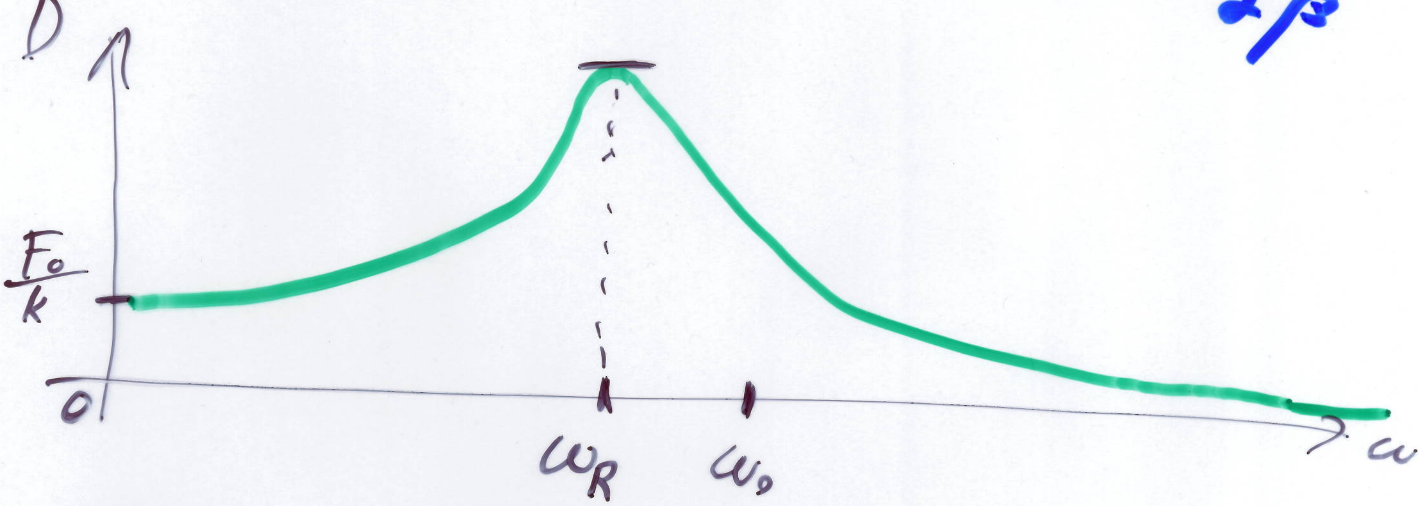
$$D = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}}$$

particular solution

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} \cos\left[\omega t - \arctan\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)\right]$$



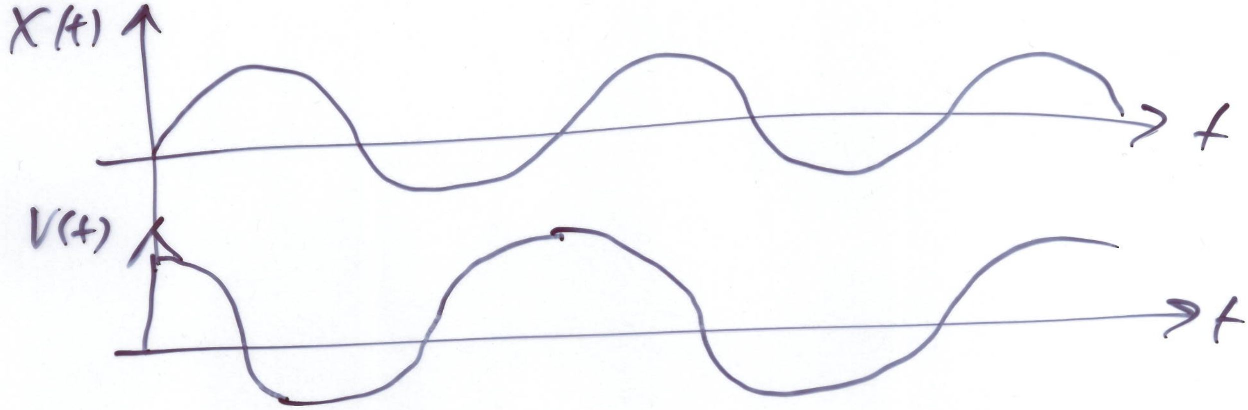
$$Q \equiv \frac{\omega_R}{2\beta}$$



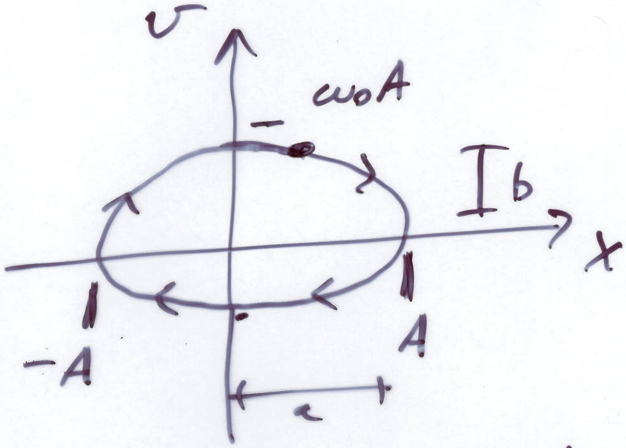
$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_R} = 0 \quad \Longrightarrow \quad \omega_R = \sqrt{\omega_0^2 - 2\beta^2} < \omega_0$$

ω_R = amplitude resonance frequency

Undamped S.H.M.



phase space



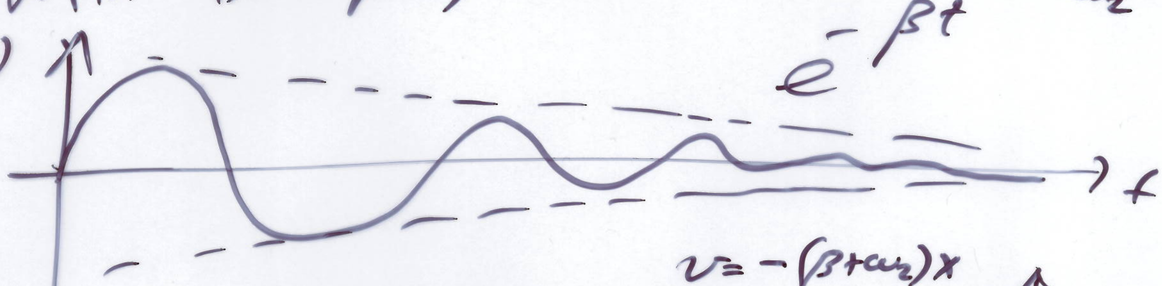
$x(t) = A\cos(\omega_0 t)$

$v(t) = -\omega_0 A \sin(\omega_0 t)$

Area ellipse : $\pi a b = \pi A(A\omega) = \dfrac{2\pi E}{\sqrt{km}}$

With Damping

$\omega_2 = \sqrt{\beta^2 - \omega_0^2}$



under-damped

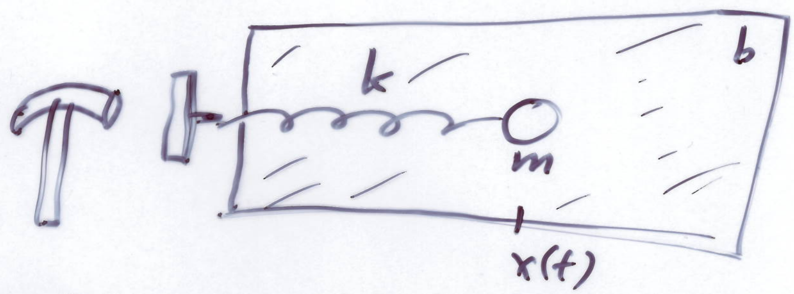
$v = -(\beta + \omega_2)x$

critically & overdamped

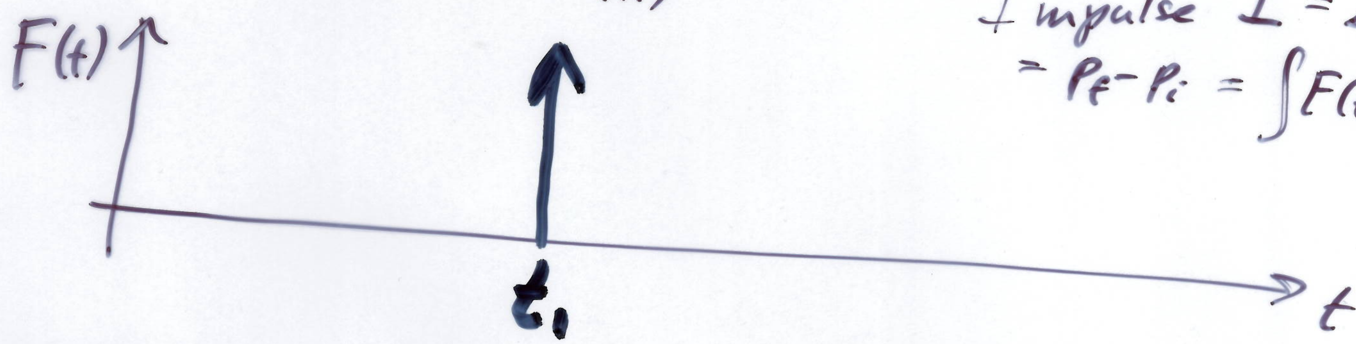
Green Functions

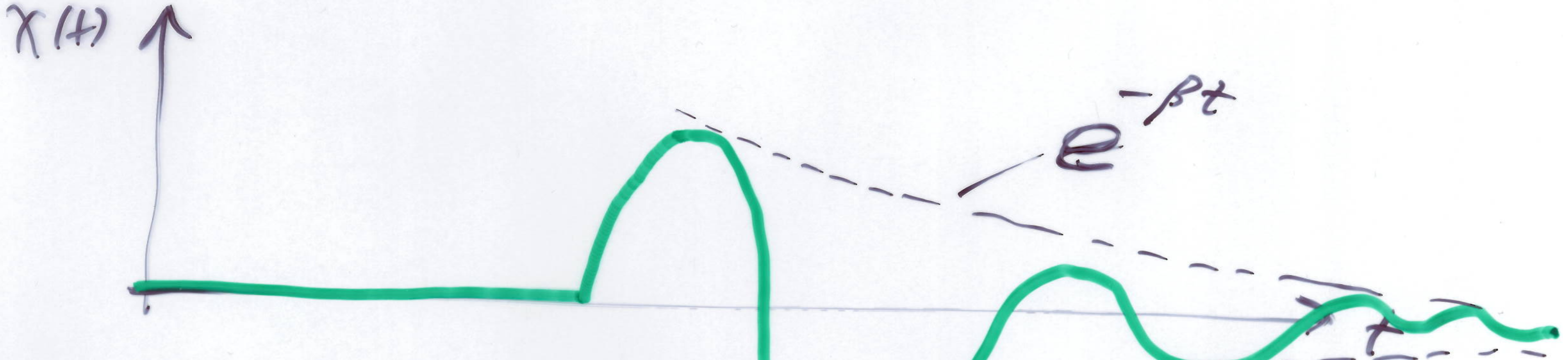
The Green function is the response to delta function forcing. The Green function is the general solution to a differential equation when the non-homogeneous driving term is impulsive.

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = \frac{F(t)}{m}$$



Impulse $I = \Delta p$
$= p_f - p_i = \int F(t)\,dt$

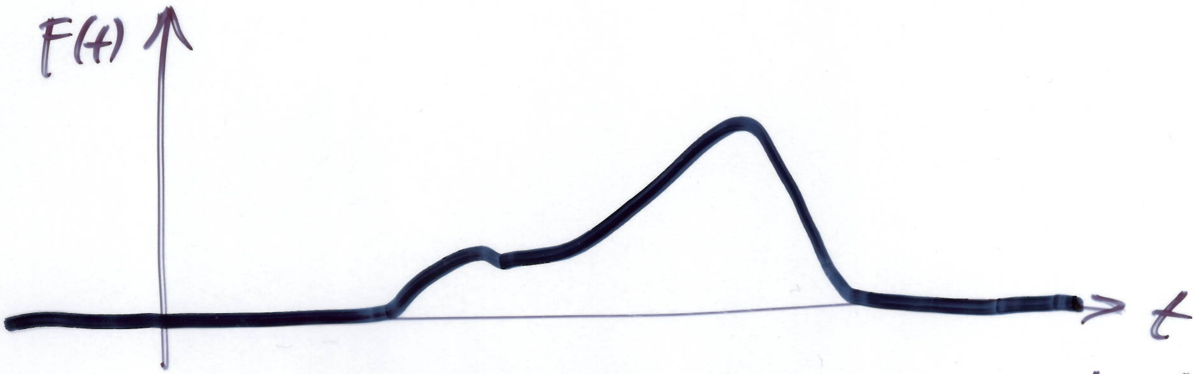




This is the Green function $G(t, t_1)$

Why is the Green function useful?

What if



How do we find $x_p(t)$ the particular solution?

An arbitrary function $F(t)$ can be decomposed as

1) sines and cosines (Fourier)
2) delta functions (Green)