e.g. 2 coupled torsion pendula

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)$$

Ansatz
$x_1 = A_1 e^{-i\omega t}$

$$V = \frac{k}{2}\left(x_1^2 + x_2^2\right) + \frac{k'}{2}\left(x - x_2\right)^2$$

$$L = T - V$$

$$\det(\underline{\underline{M}}) = \begin{vmatrix} -\omega^2 m + k + k' & -k' \\ -k' & -\omega^2 m + k + k' \end{vmatrix} = 0$$

$$\Longrightarrow (k + k' - \omega^2 m)^2 - k'^2 = 0$$

quadratic equation in ω^2

two solutions: $\underset{\substack{\text{Low} \\ \text{sym.}}}{\omega_1^2} = \frac{k}{m}$, $\underset{\substack{\text{high} \\ \text{antisym.}}}{\omega_2^2} = \frac{k + 2k'}{m}$

Substitute back in $\underline{\underline{M}}$ to get eigenvectors

$$\begin{pmatrix} k + k' - \omega_1^2 m & -k' \\ -k' & k + k' - \omega_1^2 m \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} +k' & -k' \\ -k' & +k' \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow v_1 - v_2 = 0 \qquad v_1 = v_2$$

symmetric mode

substitute ω_2^2 into $\underline{\underline{M}}$

$$\begin{pmatrix} -k' & -k' \\ -k' & -k' \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \eta_1 + \eta_2 = 0$$

$$\eta_1 = -\eta_2$$

antisymmetric mode

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intial condition

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

linear superposition

two modes oscillate at a different rate

$\implies$ after some time, the state is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Neutrino oscillation

Define $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}$ $\qquad \vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix} = \vec{A} e^{-i\omega t}$

Euler-Lagrange Equations

$$\sum_{j=1}^{N} \left(-\omega^2 T_{kj} \ddot{\eta}_j + V_{kj} \eta_j \right) = 0 \qquad k \in \{1 \ldots N\}$$

$$\underline{\underline{V}} \cdot \vec{\eta} = \underset{\underset{\lambda}{\parallel}}{\omega^2} \underline{\underline{T}} \cdot \vec{\eta} \implies \underline{\underline{V}} \cdot \vec{A}_{(p)} = \lambda_{(p)} \underline{\underline{T}} \cdot \vec{A}_{(p)}$$

label

$$\underline{\underline{V}} \cdot \vec{A}_{(p)} = \lambda_{(p)} \underline{\underline{T}} \cdot \vec{A}_{(p)}$$

This does not look like an eigenvector equation

$$\underline{\underline{M}} \cdot \vec{v} = \lambda \vec{v}$$

λ - eigenvalue
$\vec{v}$ - eigenvector

$$\left[\underline{\underline{T}}^{-1} \cdot \underline{\underline{V}}\right] \cdot \vec{A}_{(p)} = \lambda_{(p)} \vec{A}_{(p)}$$

$\lambda_{(p)}$ - eigenvalue
$\vec{A}_{(p)}$ - eigenvector

If $\lambda_{(i)} \neq \lambda_{(j)}$ then $\vec{A}_{(i)}$ is "perpendicular" to $\vec{A}_{(j)}$

$$\underline{\underline{V}} \cdot \vec{A}_{(k)} = \lambda_{(k)} \underline{\underline{T}} \cdot \vec{A}_{(k)}$$

left multiply by $\vec{A}_{(p)}$

$$\underline{\underline{V}} \cdot \vec{A}_{(p)} = \lambda_{(p)} \underline{\underline{T}} \cdot \vec{A}_{(p)}$$

left multiply by $\vec{A}_{(k)}$

subtract

$$\vec{A}_{(p)} \cdot \underline{\underline{V}} \cdot \vec{A}_{(k)} - \vec{A}_{(k)} \cdot \underline{\underline{V}} \cdot \vec{A}_{(p)} = 0$$

$\boxed{\text{Normal mode coordinates are eigenvectors } \vec{A}_{(p)}}$

component form:

$$\sum_i \sum_j A_{(p)_i} V_{ij} A_{(k)_j} - \underbrace{\sum_i \sum_j A_{(k)_i} V_{ij} A_{(p)_j}}_{\sum_i \sum_j A_{(p)_j} V_{ji} A_{(k)_i}} = 0$$

$$\underbrace{(\lambda_{(k)} - \lambda_{(p)})}_{\text{non-zero by assumption}} \underbrace{\vec{A}_{(k)} \cdot \underline{\underline{T}} \cdot \vec{A}_{(p)}}_{\text{"perpendicular"}} = 0$$