Hamiltonian ← Lagrangian $\{q_i\}$

Canonical Momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

tree-level graphs

$$L(q_i, \dot{q}_i) \rightarrow L(q_i, p_i)$$

$$H = \left(\sum_i p_i \dot{q}_i\right) - L$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad , \quad \dot{p}_i = \frac{\partial L}{\partial q_i}$$

two first order

Euler-Lagrange

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

2nd order

$$\ddot{x}[t] - N x[t] = 0 \qquad v \equiv \dot{x}$$

$$\dot{v} - x = 0$$

Chaos $\Rightarrow$ non-linearity
$\not\Leftarrow$

Simple Pendulum

$$\ddot{\theta}(t) = -\frac{g}{\ell}\sin[\theta(t)]$$

nonlinear in θ

small angle approximation

$\sin\theta \to \theta$ if $\theta \lesssim .1$ rad

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

$$\ddot{\theta}(t) = -\frac{g}{\ell}\theta(t) \Rightarrow$$

$$\theta(t) = \underline{A}\sin\left(\sqrt{\frac{g}{\ell}}\,t\right) + \underline{B}\cos\left(\sqrt{\frac{g}{\ell}}\,t\right)$$

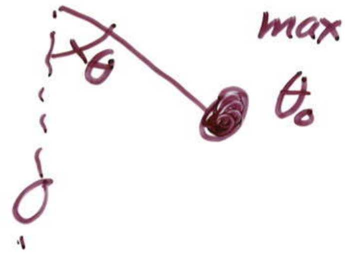
Notice if $\left[\dot{\theta}(t)\right]^2 = \frac{2g}{\ell}\left\{\cos[\theta(t)] - \cos\theta_0\right\}$

$$\frac{d}{dt}\left[\dot{\theta}(t)\right]^2 = 2\dot{\theta}(t)\ddot{\theta}(t) = -\frac{2g}{\ell}\sin[\theta(t)]\,\dot{\theta}(t)$$

$$\frac{d\theta}{dt} = \dot{\theta}(t) = \left\{\frac{2g}{\ell}\left[\cos\theta(t) - \cos\theta_0\right]\right\}^{1/2}$$

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)}} = \int_0^{t(\theta_0)} dt = \frac{1}{4} T$$

period



$$T = 4 \int_0^{\theta_0} \frac{d\theta}{\left[\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)\right]^{1/2}}$$

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$T = 2\sqrt{\frac{2\ell}{g}} \int_{\theta=0}^{\theta_0} \frac{d\theta}{\left[2\sin^2\left(\frac{\theta_0}{2}\right) - 2\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}}$$

$$\sin\left(\frac{\theta}{2}\right) \equiv \sin\left(\frac{\theta_0}{2}\right) \cdot \sin\varphi$$

$$\frac{1}{2}\cos\left(\frac{\theta}{2}\right) d\theta = \sin\left(\frac{\theta_0}{2}\right) \cos\varphi \, d\varphi$$

$$T = 4\sqrt{\frac{\ell}{g}} \int_{\varphi=0}^{\pi/2} \frac{\cos\varphi \, \sin\frac{\theta_0}{2} \, d\varphi}{\cos\left(\frac{\theta}{2}\right)\left[\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}}$$

$$T = 4\sqrt{\frac{\ell}{g}} \int_{\varphi=0}^{\pi/2} \frac{\cos\varphi \, d\varphi}{\cos\left(\frac{\theta}{2}\right)\sqrt{1 - \sin^2\varphi}}$$

$$T = 4\sqrt{\frac{\ell}{g}} \int_{\varphi=0}^{\pi/2} \frac{d\varphi}{\cos\left(\frac{\theta}{2}\right)}$$

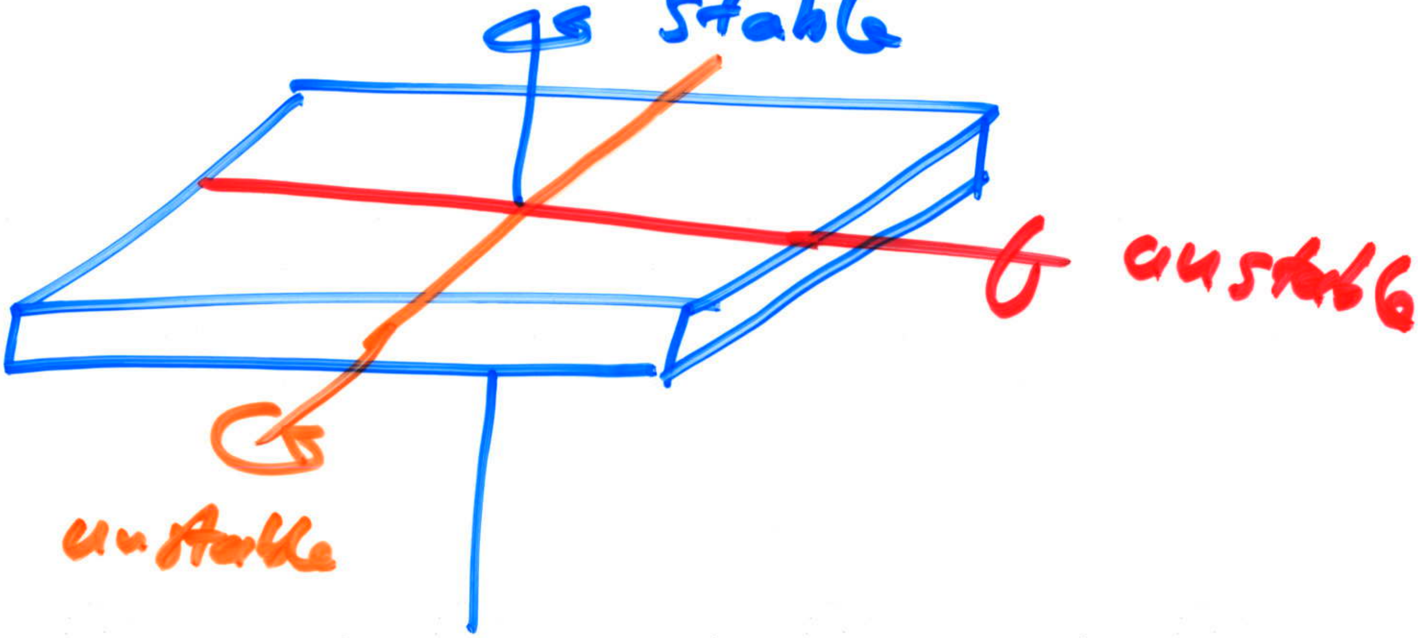
$$T = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\varphi}}$$

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{1}{2}\frac{3}{4}x^2 + \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + \cdots$$

$$T = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\pi/2} d\varphi \left[1 + \underbrace{\frac{\sin^2\frac{\theta_0}{2}\sin^2\varphi}{2}}_{\frac{\pi}{4}} + \underbrace{\frac{1}{2}\frac{3}{4}\sin^4\left(\frac{\theta_0}{2}\right)\sin^4\varphi}_{\frac{3\pi}{16}} + \cdots\right.$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1}{2}\frac{3}{4}\right)^2 \sin^4\left(\frac{\theta_0}{2}\right) + \cdots\right.$$

$$\left. + \left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\right)^2 \sin^6\left(\frac{\theta_0}{2}\right) + \cdots\right]$$


stable
unstable
unstable

where

$$\boxed{\begin{aligned} T_{\text{trans}} &= \tfrac{1}{2}\sum_{\alpha} m_\alpha V^2 = \tfrac{1}{2}MV^2 \qquad (12.6\text{a}) \\ T_{\text{rot}} &= \tfrac{1}{2}\sum_{\alpha} m_\alpha(\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2 \qquad (12.6\text{b}) \end{aligned}}$$

T_{trans} and T_{rot} designate the translational and rotational kinetic energies, respectively. Thus, the kinetic energy separates into two independent parts as mentioned in the first section of this chapter.

The rotational kinetic energy term can be evaluated by noting that

$$\begin{aligned} (\mathbf{A} \times \mathbf{B})^2 &= (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) \\ &= A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2 \end{aligned}$$

Therefore,

$$T_{\text{rot}} = \tfrac{1}{2}\sum_{\alpha} m_\alpha[\omega^2 r_\alpha^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)^2] \qquad (12.7)$$

We now express T_{rot} by making use of the components ω_i and $r_{\alpha,i}$ of the vectors $\boldsymbol{\omega}$ and $\mathbf{r}_\alpha$. We also note that $\mathbf{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ in the body system so that we can write $r_{\alpha,i} = x_{\alpha,i}$. Thus,

$$T_{\text{rot}} = \tfrac{1}{2}\sum_{\alpha} m_\alpha\left[\left(\sum_i \omega_i^2\right)\left(\sum_k x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right] \qquad (12.8)$$

Now, clearly, we can write $\omega_i = \sum_j \omega_j \delta_{ij}$, so that

$$\begin{aligned} T_{\text{rot}} &= \tfrac{1}{2}\sum_{\alpha}\sum_{i,j} m_\alpha\left[\omega_i\omega_j\delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - \omega_i\omega_j x_{\alpha,i}x_{\alpha,j}\right] \\ &= \tfrac{1}{2}\sum_{i,j}\omega_i\omega_j\sum_{\alpha} m_\alpha\left[\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right] \qquad (12.9) \end{aligned}$$

If we define the ijth element of the sum over α to be I_{ij},

$$\boxed{I_{ij} \equiv \sum_{\alpha} m_\alpha\left[\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right]} \qquad (12.10)$$

then we have

$$\boxed{T_{\text{rot}} = \tfrac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j} \qquad (12.11)$$

This equation in its most restricted form becomes

$$T_{\text{rot}} = \tfrac{1}{2}I\omega^2 \qquad (12.12)$$

where I is the (scalar) moment of inertia about the axis of rotation. This equation will be recognized as the familiar expression for the rotational kinetic energy given in elementary treatments.

The nine terms I_{ij} constitute the elements of a quantity which we designate by $\{\mathsf{I}\}$. In form, $\{\mathsf{I}\}$ is similar to a 3×3 matrix. Now, $\{\mathsf{I}\}$ is the proportionality factor between the rotational kinetic energy and the angular velocity and has the dimensions (mass) $\times$ (length)^2. Since $\{\mathsf{I}\}$ relates two quite different physical quantities, it is to be expected that $\{\mathsf{I}\}$ is a member of a somewhat higher class of functions than has heretofore been encountered. Indeed, $\{\mathsf{I}\}$ is a *tensor* and is known as the *inertia tensor*.* Note, however, that T_{rot} can be calculated, without regard to any of the special properties of tensors, by using Eq. (12.9) which completely specifies the necessary operations.

The elements of $\{\mathsf{I}\}$ can be obtained directly from Eq. (12.10). We write the elements in a 3×3 array for clarity:

$$\{\mathsf{I}\} = \begin{Bmatrix} \sum_\alpha m_\alpha(x_{\alpha,2}^2 + x_{\alpha,3}^2) & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,2} & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,1} & \sum_\alpha m_\alpha(x_{\alpha,1}^2 + x_{\alpha,3}^2) & -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_\alpha m_\alpha(x_{\alpha,1}^2 + x_{\alpha,2}^2) \end{Bmatrix} \tag{12.13}$$

The diagonal elements, I_{11}, I_{22}, and I_{33}, are called the *moments of inertia* about the x_1-, x_2-, and x_3-axes, respectively, and the negatives of the off-diagonal elements I_{12}, I_{13}, etc., are termed the *products of inertia*.† Clearly, the inertia tensor is symmetric; that is,

$$I_{ij} = I_{ji} \tag{12.14}$$

and, therefore, there are only six independent elements in $\{\mathsf{I}\}$. Furthermore, the inertia tensor is composed of additive elements; the inertia tensor for a body can be considered to be the sum of the tensors for the various portions of the body. Therefore, if we consider a body as a continuous distribution of matter with mass density $\rho = \rho(\mathbf{r})$, then

$$\boxed{I_{ij} = \int_V \rho(\mathbf{r}) \left[\delta_{ij} \sum_k x_k^2 - x_i x_j\right] dv} \tag{12.15}$$

where $dv = dx_1\, dx_2\, dx_3$ is the element of volume at the position defined by the vector $\mathbf{r}$, and where V is the volume of the body.

* The true test of a tensor is in its behavior under a coordinate transformation (see Section 12.6).

† Introduced by Huygens in 1673; Euler coined the name.

12.3 Angular Momentum

With respect to some point O that is fixed in the body coordinate system, the angular momentum of the body is

$$\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \tag{12.16}$$

The most convenient choice for the position of the point O depends on the particular problem. There are only two choices of importance: (a) If one or more points of the body are fixed (in the fixed coordinate system), O is chosen to coincide with one such point (as in the case of the rotating top, Section 12.10); (b) if no point of the body is fixed, O is chosen to be the center of mass.

Relative to the body coordinate system, the linear momentum $\mathbf{p}_\alpha$ is

$$\mathbf{p}_\alpha = m_\alpha \mathbf{v}_\alpha = m_\alpha \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

Hence, the angular momentum of the body is

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha) \tag{12.17}$$

The vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$$

can be used to express $\mathbf{L}$ as

$$\boxed{\mathbf{L} = \sum_\alpha m_\alpha [r_\alpha^2 \boldsymbol{\omega} - \mathbf{r}_\alpha(\mathbf{r}_\alpha \cdot \boldsymbol{\omega})]} \tag{12.18}$$

The same technique that was used to write T_{rot} in tensor form can now be applied here. But the angular momentum is a vector, so that for the ith component we write

$$\begin{aligned} L_i &= \sum_\alpha m_\alpha \left[\omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \sum_j x_{\alpha,j}\,\omega_j\right] \\ &= \sum_\alpha m_\alpha \sum_j \left[\omega_j\, \delta_{ij} \sum_k x_{\alpha,k}^2 - \omega_j\, x_{\alpha,i}\, x_{\alpha,j}\right] \\ &= \sum_j \omega_j \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j}\right] \end{aligned} \tag{12.19}$$

The summation over α will be recognized [cf. Eq. (12.10)] as the ijth element of the inertia tensor. Therefore,

$$\boxed{L_i = \sum_j I_{ij}\, \omega_j} \tag{12.20}$$

or, in tensor notation

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega} \tag{12.20a}$$

Thus, the inertia tensor relates a *sum* over the components of the angular velocity vector to the ith component of the angular momentum vector. This may at first seem a somewhat unexpected result; for, if we consider a rigid body for which the inertia tensor has nonvanishing off-diagonal elements, then even if $\boldsymbol{\omega}$ is directed along, say, the x_1-direction, $\boldsymbol{\omega} = (\omega_1, 0, 0)$, the angular momentum vector will be general have nonvanishing components in all three directions: $\mathbf{L} = (L_1, L_2, L_3)$. That is, the angular momentum vector does not in general have the same direction as the angular velocity vector. (It should be emphasized that this statement depends upon $I_{ij} \neq 0$ for $i \neq j$; we shall return to this point in the next section.)

As an example of a situation in which $\boldsymbol{\omega}$ and $\mathbf{L}$ are not co-linear, consider the rotating dumbbell in Fig. 12-2. (The shaft connecting m_1 and m_2 is

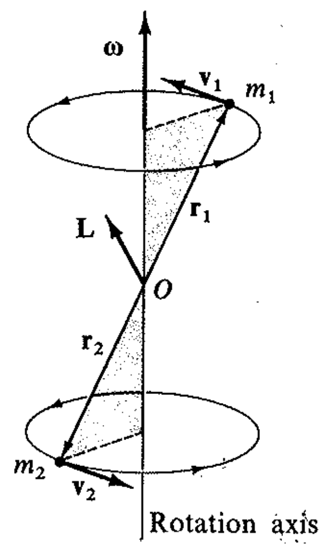


FIG. 12-2

considered to be weightless and extensionless.) The relation connecting $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$, and $\boldsymbol{\omega}$ is

$$\mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

and the relation connecting $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$, and $\mathbf{L}$ is

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha$$

Then, clearly, $\boldsymbol{\omega}$ is directed along the axis of rotation, while $\mathbf{L}$ is perpendicular to the line connecting m_1 and m_2.

We note, for this example, that the angular momentum vector $\mathbf{L}$ does not remain constant in time, but rotates with an angular velocity ω in such a way that it traces out a cone whose axis is the axis of rotation. Therefore, $\dot{\mathbf{L}} \neq 0$; but Eq. (2.11) states that

$$\dot{\mathbf{L}} = \mathbf{N}$$

where $\mathbf{N}$ is the torque applied to the body. Thus, in order to keep the dumbbell rotating as in Fig. 12-2, a torque must be constantly applied.

We can obtain another result from Eq. (12.20) by multiplying L_i by $\frac{1}{2}\omega_i$ and summing over i:

$$\frac{1}{2}\sum_i \omega_i L_i = \frac{1}{2}\sum_{i,j} I_{ij}\,\omega_i\,\omega_j = T_{\text{rot}} \tag{12.21}$$

where the second equality is just Eq. (12.11). Thus,

$$\boxed{T_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}} \tag{12.21a}$$

Equations (12.20a) and (12.21a) illustrate two important properties of tensors. The product of a tensor and a vector yields a vector, as in

$$\mathbf{L} = \{\mathsf{I}\}\cdot\boldsymbol{\omega}$$

and the product of a tensor and two vectors yields a scalar, as in

$$T_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \tfrac{1}{2}\boldsymbol{\omega}\cdot\{\mathsf{I}\}\cdot\boldsymbol{\omega}$$

We shall not, however, have occasion to use here tensor equations written in the above form, but will always use the summation (or integral) expressions as in Eqs. (12.11), (12.15), (12.20), etc.

12.4 Principal Axes of Inertia*

It is clear that a considerable simplification in the expressions for T and $\mathbf{L}$ would result if the inertia tensor consisted only of diagonal elements. If we could write

$$I_{ij} = I_i\,\delta_{ij} \tag{12.22}$$

then the inertia tensor would be

$$\{\mathsf{I}\} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix} \tag{12.23}$$

* Discovered by Euler, 1750.

(a) Diagonalization may be accomplished by an appropriate rotation of axes, i.e., a similarity transformation.

(b) The eigenvalues* are obtained as roots of the secular determinant and are real.

(c) The eigenvectors* are real and orthogonal.

12.7 The Eulerian Angles

The transformation from one coordinate system to another can be represented by a matrix equation of the form

$$\mathsf{x} = \lambda \mathsf{x}'$$

If we identify the fixed system with x' and the body system with x, then the rotation matrix λ completely describes the relative orientation of the two systems. Now, the rotation matrix λ contains three independent angles. There are many possible choices for these angles; for our purposes we will find it convenient to use the so called *Eulerian angles*,† φ, θ, and ψ.

The Eulerian angles are generated in the following series of rotations which take the x_i' system into the x_i system‡:

(1) The first rotation is counterclockwise through an angle φ about the x_3'-axis, as shown in Fig. 12-6a to transform the x_i' into the x_i''. Since the rotation takes place in the x_1'-x_2' plane, the transformation matrix is

$$\lambda_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.62}$$

(2) The second rotation is counterclockwise through an angle θ about the x_1''-axis, as shown in Fig. 12-6b, to transform the x_i'' into the x_i'''. Since the rotation is now in the x_2''-x_3'' plane, the transformation matrix is

$$\lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{12.63}$$

(3) The third rotation is counterclockwise through an angle ψ about the x_3'''-axis, as shown in Fig. 12-6c, to transform the x_i''' into the x_i. The

* The terms "eigenvalues" and "eigenvectors" are the generic names of the quantities which, in the case of the inertia tensor, are the principal moments and the principal axes, respectively. We shall encounter these terms again in the discussion of small oscillations in Chapter 13.

† The rotation scheme of Euler was first published in 1776.

‡ It should be noted that the designations of the Euler angles and even the manner in which they are generated are not universally agreed upon. Therefore, some care must be taken in any comparison of results from different sources. The notation used here is that most commonly found in modern texts.

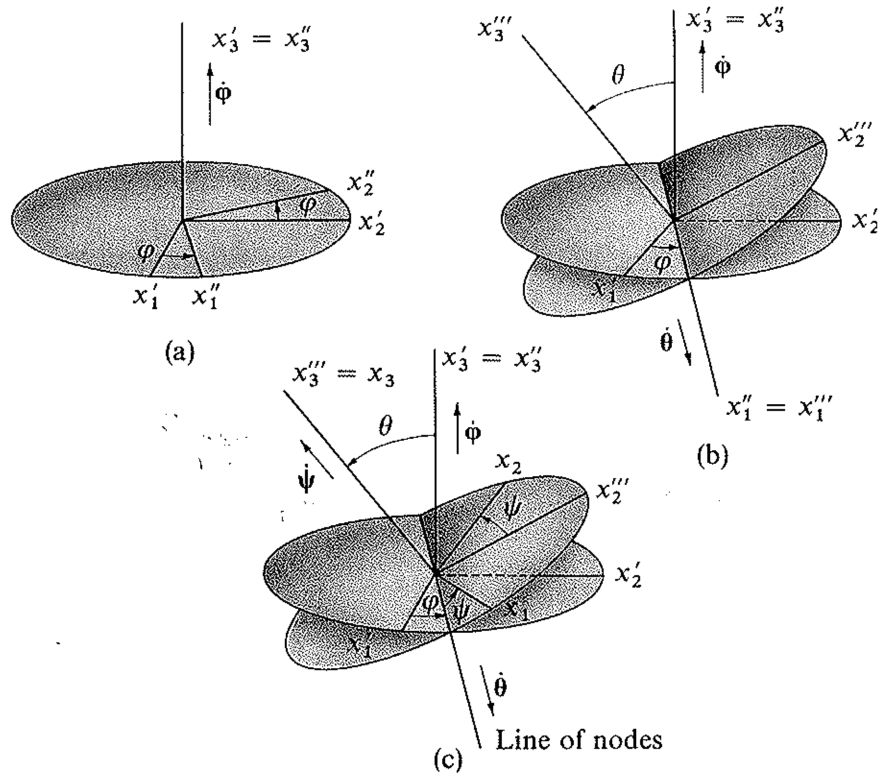


FIG. 12-6

transformation matrix is

$$\lambda_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.64}$$

The line which is common to the planes containing the x_1- and x_2-axes and the x'_1- and x'_2-axes is called the *line of nodes*.

The complete transformation from the x'_i system to the x_i system is given by the rotation matrix λ:

$$\lambda = \lambda_\psi \lambda_\theta \lambda_\varphi \tag{12.65}$$

The components of this matrix are:

$$\left.\begin{aligned}
\lambda_{11} &= \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi \\
\lambda_{21} &= -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi \\
\lambda_{31} &= \sin\theta\sin\varphi \\
\lambda_{12} &= \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi \\
\lambda_{22} &= -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi \\
\lambda_{32} &= -\sin\theta\cos\varphi \\
\lambda_{13} &= \sin\psi\sin\theta \\
\lambda_{23} &= \cos\psi\sin\theta \\
\lambda_{33} &= \cos\theta
\end{aligned}\right\} \tag{12.66}$$

(The components λ_{ij} are off-set above to assist in the visualization of the complete $\boldsymbol{\lambda}$-matrix.)

Since it is possible to associate a vector with an infinitesimal rotation, we can associate the time derivatives of these rotation angles with the components of the angular velocity vector $\boldsymbol{\omega}$. Thus,

$$\left.\begin{aligned} \omega_\varphi &= \dot{\varphi} \\ \omega_\theta &= \dot{\theta} \\ \omega_\psi &= \dot{\psi} \end{aligned}\right\} \tag{12.67}$$

The rigid-body equations of motion are most conveniently expressed in the body coordinate system (i.e., the x_i system), and therefore we must express the components of $\boldsymbol{\omega}$ is this system. We note that, in Fig. 12-6, the angular velocities $\dot{\boldsymbol{\varphi}}$, $\dot{\boldsymbol{\theta}}$, and $\dot{\boldsymbol{\psi}}$ are directed along the following axes:

$\dot{\boldsymbol{\varphi}}$ along the x'_3 (fixed) axis
$\dot{\boldsymbol{\theta}}$ along the line of nodes
$\dot{\boldsymbol{\psi}}$ along the x_3 (body) axis

The components of these angular velocities along the body coordinate axes are:

$$\left.\begin{aligned} \dot{\varphi}_1 &= \dot{\varphi} \sin\theta \sin\psi \\ \dot{\varphi}_2 &= \dot{\varphi} \sin\theta \cos\psi \\ \dot{\varphi}_3 &= \dot{\varphi} \cos\theta \end{aligned}\right\} \tag{12.68a}$$

$$\left.\begin{aligned} \dot{\theta}_1 &= \dot{\theta} \cos\psi \\ \dot{\theta}_2 &= -\dot{\theta} \sin\psi \\ \dot{\theta}_3 &= 0 \end{aligned}\right\} \tag{12.68b}$$

$$\left.\begin{aligned} \dot{\psi}_1 &= 0 \\ \dot{\psi}_2 &= 0 \\ \dot{\psi}_3 &= \dot{\psi} \end{aligned}\right\} \tag{12.68c}$$

Collecting the individual components of $\boldsymbol{\omega}$, we have, finally,

$$\boxed{\begin{aligned} \omega_1 &= \dot{\varphi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \omega_2 &= \dot{\varphi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\varphi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \omega_3 &= \dot{\varphi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\varphi} \cos\theta + \dot{\psi} \end{aligned}} \tag{12.69}$$

These relations will be of use later in expressing the components of the angular momentum in the body coordinate system.

12.8 Euler's Equations for a Rigid Body

Let us first consider the force-free motion of a rigid body. In such a case, the potential energy U vanishes and the Lagrangian L becomes identical with the rotational kinetic energy T.* If we choose the x_i-axes to correspond to the principal axes of the body, then from Eq. (12.24b) we have

$$T = \tfrac{1}{2}\sum_i I_i \omega_i^2 \tag{12.70}$$

If we choose the Eulerian angles as the generalized coordinates, then Lagrange's equation for the coordinate ψ is

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} = 0 \tag{12.71}$$

which can be expressed as

$$\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} = 0 \tag{12.72}$$

If we differentiate the components of $\boldsymbol{\omega}$ [Eqs. (12.69)] with respect to ψ and $\dot{\psi}$ we have

$$\left.\begin{aligned} \frac{\partial \omega_1}{\partial \psi} &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \omega_2 \\ \frac{\partial \omega_2}{\partial \psi} &= -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi = -\omega_1 \\ \frac{d\omega_3}{\partial \psi} &= 0 \end{aligned}\right\} \tag{12.73}$$

and

$$\left.\begin{aligned} \frac{\partial \omega_1}{\partial \dot{\psi}} &= \frac{\partial \omega_2}{\partial \dot{\psi}} = 0 \\ \frac{\partial \omega_3}{\partial \dot{\psi}} &= 1 \end{aligned}\right\} \tag{12.74}$$

From Eq. (12.70) we also have

$$\frac{\partial T}{\partial \omega_i} = I_i \omega_i \tag{12.75}$$

* Since the motion is force-free, the translational kinetic energy is unimportant for our purposes here. (We can always transform to a coordinate system in which the center of mass of the body is at rest.)

Therefore, Eq. (12.72) becomes

$$I_1\omega_1\omega_2 + I_2\,\omega_2(-\omega_1) - \frac{d}{dt}\,I_3\,\omega_3 = 0$$

or,

$$(I_1 - I_2)\omega_1\omega_2 - I_3\,\dot{\omega}_3 = 0 \tag{12.76}$$

Since the designation of any particular principal axis as the x_3-axis is entirely arbitrary, Eq. (12.76) can be permuted to obtain relations for $\dot{\omega}_1$ and $\dot{\omega}_2$. By making use of the permutation symbol, we can write, in general,

$$\boxed{(I_i - I_j)\omega_i\,\omega_j - \sum_k I_k\,\dot{\omega}_k\,\varepsilon_{ijk} = 0} \tag{12.77}$$

The three equations represented by Eq. (12.77) are called *Euler's equations* for the case of force-free motion.* It must be noted that although Eq. (12.76) for $\dot{\omega}_3$ is indeed the Lagrange equation for the coordinate ψ, the Euler equations for $\dot{\omega}_1$ and $\dot{\omega}_2$, which can be obtained from Eq. (12.77), are *not* the Lagrange equations for θ and φ.

In order to obtain Euler's equations for the case of motion in a force field, we may start with the fundamental relation for the torque $\mathbf{N}$ [cf. Eq. (2.11)]:

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \mathbf{N} \tag{12.78}$$

where the designation "fixed" has been explicitly appended to $\dot{\mathbf{L}}$ since this relation is derived from Newton's equation and is therefore valid only in an inertial frame of reference. From Eq. (11.7) we have

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} \tag{12.79}$$

or,

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N} \tag{12.80}$$

The component of this equation along the x_3-axis (note that this is a *body* axis) is

$$\dot{L}_3 + \omega_1 L_2 - \omega_2 L_1 = N_3 \tag{12.81}$$

* Leonard Euler, 1758.

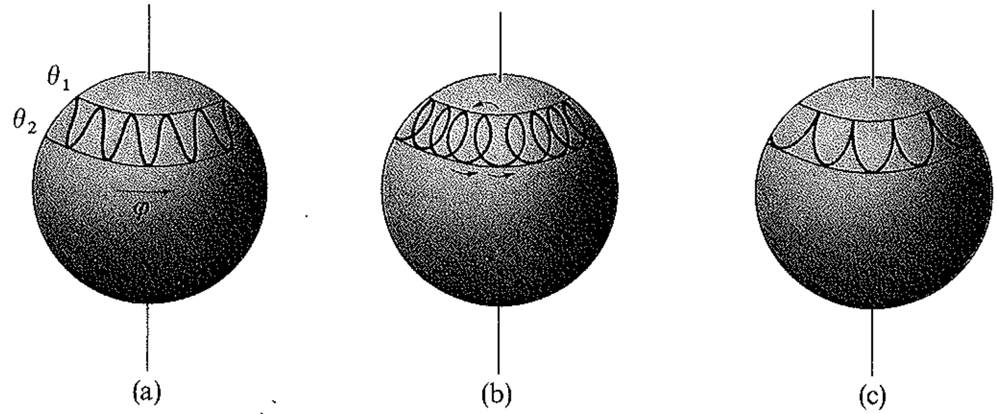


FIG. **12-11**

and Fig. 12-11c shows the resulting cusplike motion. It is just this case that corresponds to the usual method of starting a top. First, the top is set to spinning around its axis, then it is given a certain initial tilt and released. Thus, the initial conditions are $\theta = \theta_1$, and $\dot{\theta} = 0 = \dot{\varphi}$. Since the first motion of the top is to begin to fall in the gravitational field, the conditions are exactly those of Fig. 12-11c, and the cusplike motion ensues. Figures 12-11a and b correspond to the motion in the event that there is an initial angular velocity $\dot{\varphi}$ either in the direction of or opposite to the direction of precession.

12.11 The Stability of Rigid-Body Rotations*

We now consider a rigid body which is undergoing force-free rotation around one of its principal axes and inquire whether such motion is stable, "Stability" here means, as before (see Section 8.11), that if a small perturbation is applied to the system, the motion will either return to its former mode or will perform small oscillations about it.

We choose for our discussion a general rigid body for which all of the principal moments of inertia are distinct, and we label them in such a way that $I_3 > I_2 > I_1$. We let the body axes coincide with the principal axes, and we start with the body rotating around the x_1-axis, i.e., around the principal axis associated with the moment of inertia I_1. Then,

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 \tag{12.123}$$

If we apply a small perturbation, the angular velocity vector will assume the form

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 \tag{12.124}$$

* This problem was first treated by Euler in 1749.

where λ and μ are small quantities and correspond to the parameters which have been used previously in other perturbation expansions. (λ and μ are sufficiently small so that their product can be neglected compared to all other quantities of interest to the discussion.)

The Euler equations become [see Eq. (12.77)]:

$$\left.\begin{aligned}(I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 &= 0\\ (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} &= 0\\ (I_1 - I_2)\lambda\omega_1 - I_3\dot{\mu} &= 0\end{aligned}\right\} \tag{12.125}$$

Since $\lambda\mu \approx 0$, the first of these equations requires $\dot{\omega}_1 = 0$, or $\omega_1 = \text{const.}$ Solving the other two equations for $\dot{\lambda}$ and $\dot{\mu}$, we find

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\,\omega_1\right)\mu \tag{12.126}$$

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3}\,\omega_1\right)\lambda \tag{12.127}$$

where the terms in parentheses are both constants. These are coupled equations, but they cannot be solved by the method employed in Section 12.9 since the constants in the two equations are different. The solution can be obtained by first differentiating the equation for $\dot{\lambda}$:

$$\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\,\omega_1\right)\dot{\mu} \tag{12.128}$$

The expression for $\dot{\mu}$ can now be substituted in this equation:

$$\ddot{\lambda} + \left(\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\,\omega_1^2\right)\lambda = 0 \tag{12.129}$$

The solution to this equation is

$$\lambda(t) = Ae^{i\Omega_{1\lambda}t} + Be^{-i\Omega_{1\lambda}t} \tag{12.130}$$

where

$$\Omega_{1\lambda} \equiv \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{12.131}$$

and where the subscripts 1 and λ designate that we are considering the solution for λ when the rotation is around the x_1-axis.

By hypothesis, $I_1 < I_3$ and $I_1 < I_2$, so that $\Omega_{1\lambda}$ is real. Therefore, the solution for $\lambda(t)$ represents oscillatory motion with a frequency $\Omega_{1\lambda}$. We can similarly investigate $\mu(t)$ with the result that $\Omega_{1\mu} = \Omega_{1\lambda} \equiv \Omega_1$. Thus, the small perturbations introduced by forcing small x_2- and x_3-components on $\boldsymbol{\omega}$ do not increase with time, but oscillate around the equilibrium values $\lambda = 0$ and $\mu = 0$. Consequently, the rotation around the x_1-axis is stable.

If we consider rotations around the x_2- and x_3-axes, expressions for Ω_2 and Ω_3 can be obtained from Eq. (12.131) by permutation:

$$\Omega_1 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{12.132a}$$

$$\Omega_2 = \omega_2\sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}} \tag{12.132b}$$

$$\Omega_3 = \omega_3\sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}} \tag{12.132c}$$

But since $I_1 < I_2 < I_3$, we have

$$\Omega_1, \quad \Omega_3 \quad \text{real}; \qquad \Omega_2 \quad \text{imaginary}$$

Thus, when the rotation takes place around either the x_1- or x_3-axes, the perturbation produces oscillatory motion and the rotation is stable. When the rotation takes place around x_2, however, the fact that Ω_2 is imaginary results in the perturbation increasing with time without limit; such motion is unstable.

Since we have assumed a completely arbitrary rigid body for this discussion, we conclude that rotation around the principal axis corresponding to either the greatest or smallest moment of inertia is stable, while rotation around the principal axis corresponding to the intermediate moment is unstable. This effect can be easily demonstrated with, say, a book (which is kept closed by tape or a rubber band). If the book is tossed into the air with an angular velocity around one of the principal axes, the motion will be unstable for rotation around the intermediate axis and stable for the other two axes.

In the event that two of the moments of inertia are equal ($I_1 = I_2$, say), then the coefficient of λ in Eq. (12.127) vanishes, and we have $\dot{\mu} = 0$ or $\mu(t) = \text{const}$. Therefore, Eq. (12.126) for λ can be integrated to yield

$$\lambda(t) = C + Dt \tag{12.133}$$

and the perturbation increases linearly with the time; the motion around the x_1-axis is therefore unstable. We find a similar result for motion around the x_2-axis. There is stability only for the x_3-axis, independent of whether I_3 is greater or less than $I_1 = I_2$.

Suggested References

Rigid-body dynamics is a topic discussed in almost every mechanics text. Introductory accounts are given, for example, by Fowles (Fo62, Chapter 9) and by Lindsay (Li61, Chapter 8). At a slightly more advanced level are the treatments of Becker (Be54, Chapter 12), Constant (Co54, Chapter 9), Slater and Frank (Sl47, Chapter 6), and Sommerfeld (So50, Chapter 4).