• <u>def</u> rank $\underline{n}$ tensor – a quantity that changes like the exterior product of $\underline{n}$ position vectors

T_{ij} changes under rotations like $r_i r_j$ (or $v_i v_j$)
↑ rank 2 $i,j = 1,\dots,3$ (d in general)

T_{ij} can be represented as a matrix, <u>but</u> not all matrices are rank 2 tensors. In particular, the transformation matrix that relates $\vec{r}\,'$ to $\vec{r}$ is <u>not</u> a tensor.

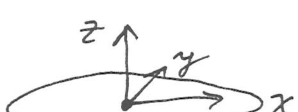
e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2x2 identity matrix is <u>not</u> a rank 2 tensor.

proof! $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_1^2 & r_1 r_2 \\ r_2 r_1 & r_2^2 \end{pmatrix}$ $r_1^2 = 1 \Rightarrow r_1 \neq 0$
$r_2^2 = 1 \Rightarrow r_2 \neq 0$
but $r_1 r_2 = 0$. Impossible.

e.g. rank 2 tensors: θ_{ij} energy-momentum tensor
I_{ij} moment of inertia tensor
(by the way, I = mass, $I_i \propto$ displacement vector of center of mass) $I_i = m \vec{r}_{cm}$
~ mean   ~ standard deviation
ϵ_{ij} dielectric tensor

T_{ijk} changes like $r_i r_j r_k$
↑ rank 3

Hold it! I thought that the moment of inertia about the center of mass for a disk (for example) was $\frac{1}{2} m R^2$. How is this a tensor?



$I_{zz} = \frac{1}{2} m R^2$ $I_{xx} = I_{yy} \neq 0$

$I_{xy} = 0$; $I_{yz} = 0$; $I_{xz} = 0$

Examples:

scalar — rank 0 — no indices — does not transform under coordinate rotations

$$I = \int \rho \, dV = \text{mass} = m$$

volume

density

not standard notation

---

vector — rank 1 — one index — transforms like X_i

$$I_i = \int x_i \rho \, dV = m(X_{cm})_i$$

$$\vec{I} = \int \vec{X} \rho \, dV = \int \vec{r} \rho \, dV = m \vec{X}_{cm}$$

not standard notation

center of mass location

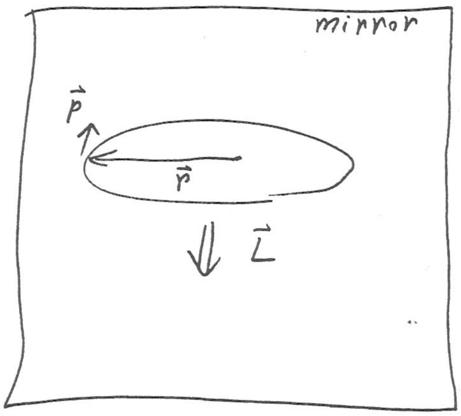
---

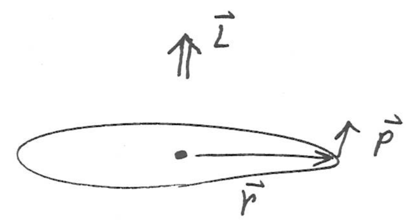
rank 2 tensor — two indices — transforms like $X_i X_j$

$$I_{ij} = \int X_i X_j \rho \, dV$$

standard notation for moment of inertia tensor

Reflections



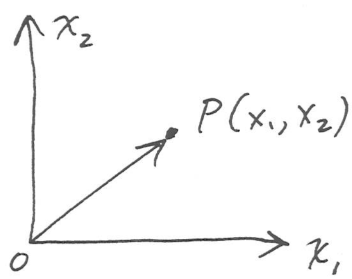


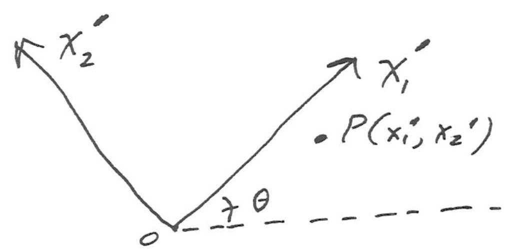
$\vec{r}, \vec{p}$ are vectors (polar vectors)
$\vec{L} = \vec{r} \times \vec{p}$ is a pseudo vector (axial vector)

How does $\vec{r}$ change under rotations?

Special case: rotations in 2 dimensions (can't rotate in 1-d)
Later we will generalize to higher dimensions.

R_θ is the rotation transformation by angle θ (one parameter)





Passive transformation: P stays fixed, axes are rotated by angle θ counterclockwise about origin.

Active transformation: axes stay fixed, P is rotated by angle θ clockwise about origin.

R_θ preserves: the lengths of vectors
the angle between vectors

such a transformation is called "orthogonal."

Two-dimensional rotations form a mathematical "group" $U(1)$

Properties: 1) $R_{\theta_2} R_{\theta_1} (\vec{x}) = R_{\theta_3} (\vec{x})$ $\qquad \theta_3 = \theta_1 + \theta_2$

two successive rotations are equivalent to a third rotation,

2) $R_0 = \mathbb{I}$ identity. $\quad R_0 \vec{x} = \vec{x} \qquad R_{2\pi n} \vec{x} = \vec{x}$

n integer

3) $R_{-\theta} R_\theta (\vec{x}) = \mathbb{I} \vec{x} = \vec{x}$

every element of the group has an inverse,

The order of 2-d rotations is irrelevant. This kind of group is called "abelian".

$$R_{\theta_2} R_{\theta_1} (\vec{x}) = R_{\theta_1} R_{\theta_2} (\vec{x})$$

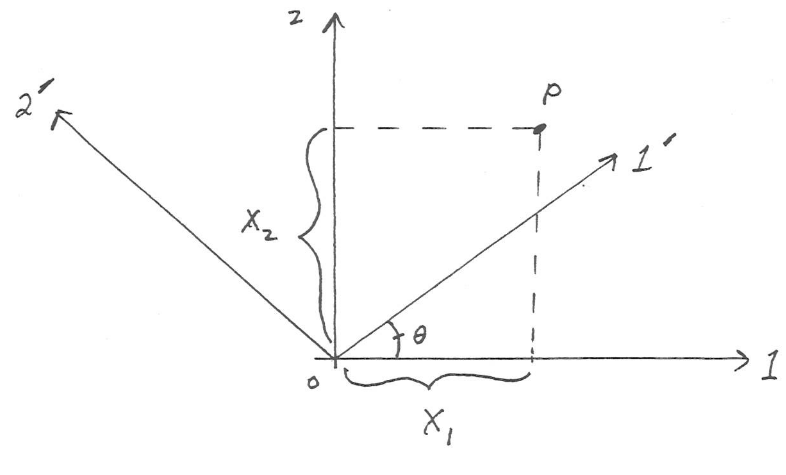
or

$$R_{-\theta_2} R_{-\theta_1} R_{\theta_2} R_{\theta_1} (\vec{x}) = \vec{x}$$

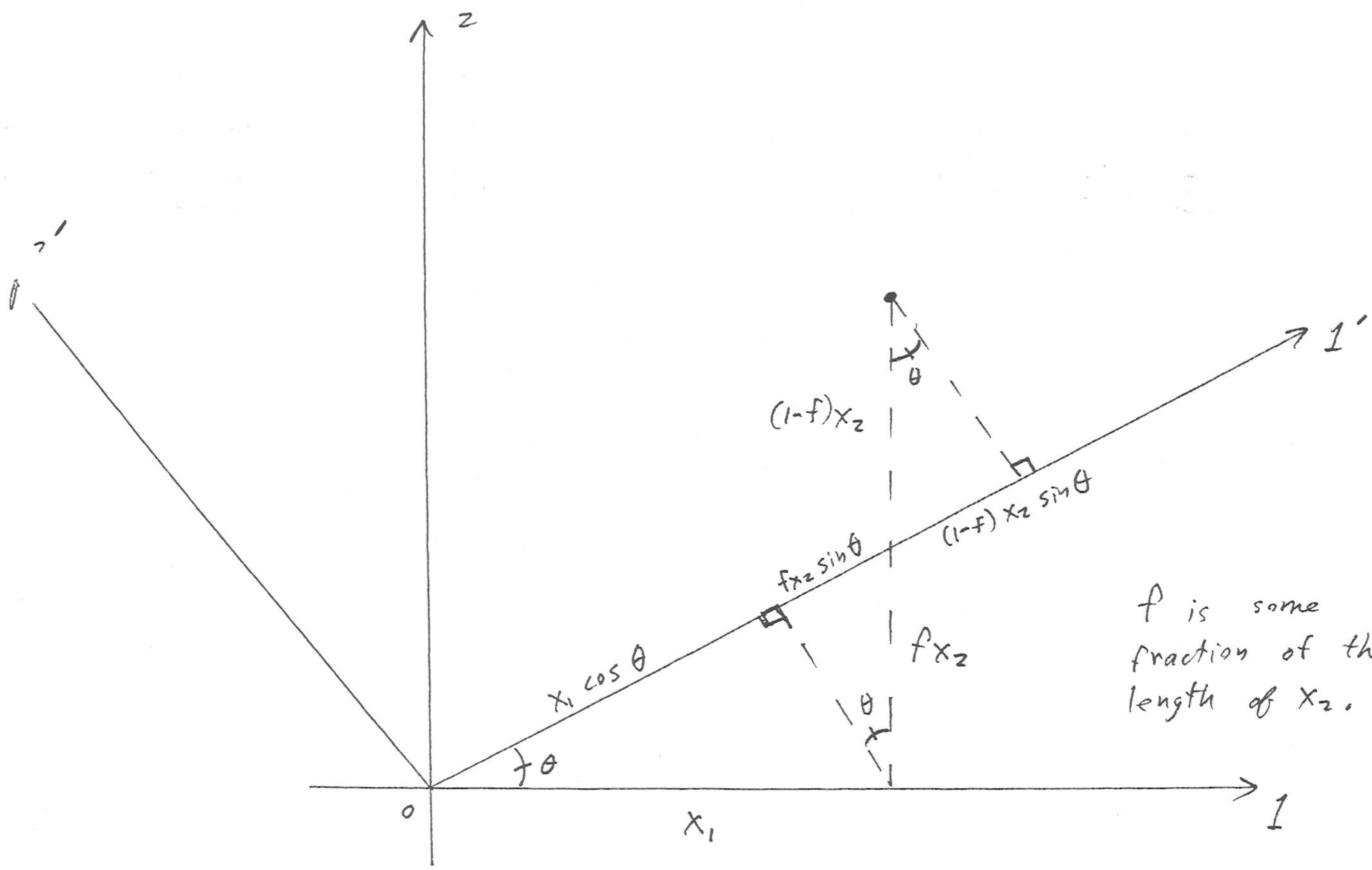
Not true for 3-d rotations (e.g. rotations of book).

The world is non-abelian: You cook your food, then you eat it!

2-dimensional rotations



Passive Transformation: leave P fixed and rotate axes by angle θ counterclockwise.



f is some fraction of the length of X_2.

$$X_1' = X_1 \cos\theta + f X_2 \sin\theta + (1-f) X_2 \sin\theta$$

now you see the fraction f is irrelevant

$$X_1' = X_1 \cos\theta + X_2 \sin\theta$$

By a similar geometrical construction, you can find x_2':

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$
$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

Rewrite these equations in matrix form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ check this!

And remember how to multiply matrices

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$$

$$\begin{array}{cc} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \end{array}$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21} \qquad c_{12} = a_{11} b_{12} + a_{12} b_{22}$$
$$c_{21} = a_{21} b_{11} + a_{22} b_{21} \qquad c_{22} = a_{21} b_{12} + a_{22} b_{22}$$

One more time in condensed form

$\vec{x}' = \underline{\underline{\lambda}}\, \vec{x}$ where $\underline{\underline{\lambda}}$ is the 2-dim transformation matrix with components

$$\underline{\underline{\lambda}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \qquad \lambda_{11} = \cos\theta \qquad \lambda_{12} = \sin\theta$$
$$\lambda_{21} = -\sin\theta \qquad \lambda_{22} = \cos\theta$$

($\underline{\underline{\lambda}}$ is what we called R_θ last time.)

It is often useful to write the transformation in

<u>Index Notation</u>

ith row
jth column
entry of λ matrix

$$\vec{x}' = \underset{\approx}{\lambda}\, \vec{x} \qquad \Longleftrightarrow \qquad x_i' = \sum_{j=1}^{2} \lambda_{ij}\, x_j$$

For example, when $i=1$ this gives

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2 = \cos\theta\, x_1 + \sin\theta\, x_2 .$$

which is the same equation we had previously.

Two things to notice:

① j is a "dummy" index — because j is summed over, one can call it by <u>any</u> name, k for example

$$x_i' = \sum_{k=1}^{2} \lambda_{ik}\, x_k$$

② In matrix multiplication the sum occurs over <u>adjacent</u> indices. In the example above, the k's touch each other.

One can solve the transformation equations

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$
$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

(2 equations in 2 unknowns)

for the unprimed variables using algebra to get

$$x_1 = x_1' \cos\theta - x_2' \sin\theta$$
$$x_2 = x_1' \sin\theta + x_2' \cos\theta$$

But this is the hard way. The easy way is to notice that primed and unprimed are arbitrary labels for the axes and it doesn't matter which is which. If you change

$$\text{primed} \longleftrightarrow \text{unprimed}$$

then you must also change $\theta \longleftrightarrow -\theta$.

Write the inverse transformation in matrix form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \qquad \text{or} \qquad \vec{x} = \underline{\underline{\mu}}\,\vec{x}'$$

where the matrix $\underline{\underline{\mu}}$ is the <u>transpose</u> of $\underline{\underline{\lambda}}$

$$\underline{\underline{\mu}} = \underline{\underline{\lambda}}^T \qquad \text{(switch rows and columns)}$$

$$\mu_{ij} = \lambda_{ji}$$

Now we have $\vec{x} = \underline{\underline{\mu}}\,\vec{x}'$ from above and $\vec{x}' = \underline{\underline{\lambda}}\,\vec{x}$ from before. Combining these

$$\vec{x} = \underline{\underline{\mu}}(\vec{x}') = \underline{\underline{\mu}}(\underline{\underline{\lambda}}\,\vec{x}) = (\underline{\underline{\mu}}\,\underline{\underline{\lambda}})\,\vec{x}$$

this can only be true if

$$\underline{\underline{\mu}}\,\underline{\underline{\lambda}} = \underline{\underline{I}} \quad (2\times 2 \text{ identity matrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{I}} \qquad \text{also} \qquad \underline{\underline{\lambda}}\,\underline{\underline{\lambda}}^T = \underline{\underline{I}}$$

or

$$\lambda^{-1} = \lambda^T \qquad \text{(inverse = transpose)}$$

A matrix $\underline{\underline{\lambda}}$ that obeys $\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{\mathbb{I}}}$ is called "orthogonal." That means the transformation preserves:

1) Lengths of vectors
2) Angles between vectors

Let's write the orthogonality relation in index notation for practice:

$$\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{\mathbb{I}}} \iff \sum_{j=1}^{2} (\lambda^T)_{ij} \lambda_{jk} = \mathbb{I}_{ik}$$

adjacent index
is summed over
in matrix multiplication

Now two alterations:

$$(\lambda^T)_{ij} = \lambda_{ji} \qquad \text{transpose switches } i \leftrightarrow j$$

$\mathbb{I}_{ik}$ is the element in the ith row, jth column of the identity matrix.

1 if $i = k$ and 0 if $i \neq k$

There is a symbol for this

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \qquad \text{The Kronecker delta}$$

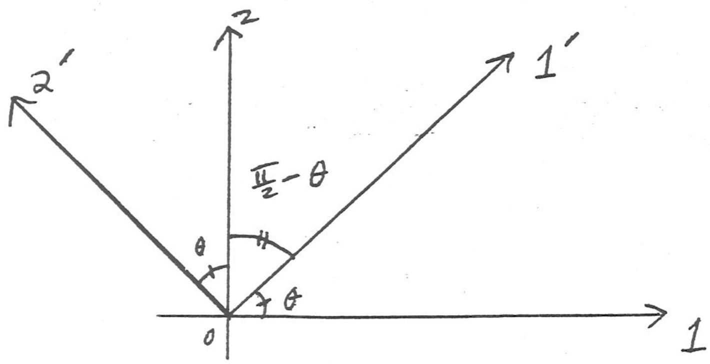
$$\sum_{j=1}^{2} \lambda_{ji} \lambda_{jk} = \delta_{ik} \qquad \text{orthogonality}$$

No longer matrix multiplication since the summed indices don't touch.

Direction Cosines

$$\underline{\underline{\lambda}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\left(\frac{\pi}{2}-\theta\right) \\ \cos\left(\frac{\pi}{2}+\theta\right) & \cos\theta \end{pmatrix}$$

λ_{11}	is the cosine of the angle between	$1'$ and 1 axes	
λ_{12}	"	$1'$	2
λ_{21}	"	$2'$	1
λ_{22}	"	$2'$	2



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the orthogonality condition imposes 3 constraints on the 4 elements of the $\underline{\underline{\lambda}}$ matrix, leaving one "degree of freedom."

$$\sum_{j=1}^{2} \lambda_{ji}\,\lambda_{jk} = \delta_{ik}$$

① $\lambda_{11}\lambda_{11} + \lambda_{21}\lambda_{21} = 1$ $\quad (i=1,\ k=1)$

② $\lambda_{12}\lambda_{12} + \lambda_{22}\lambda_{22} = 1$ $\quad (i=2,\ k=2)$

③ $\lambda_{11}\lambda_{12} + \lambda_{21}\lambda_{22} = 0$ $\quad (i=1,\ k=2 \ \underline{\underline{\text{or}}}\ i=2,\ k=1)$

same constraint

Generalize to 3 dimensions

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2 + \lambda_{13} x_3$$

$$x_2' = \lambda_{21} x_1 + \lambda_{22} x_2 + \lambda_{23} x_3$$

$$x_3' = \lambda_{31} x_1 + \lambda_{32} x_2 + \lambda_{33} x_3$$

In matrix form:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In condensed form: $\vec{x}' = \underline{\underline{\lambda}} \, \vec{x}$

In index notation: $x_p' = \sum_{a=1}^{3} \lambda_{pa} x_a$

The 3-dimensional rotation matrix is also orthogonal:

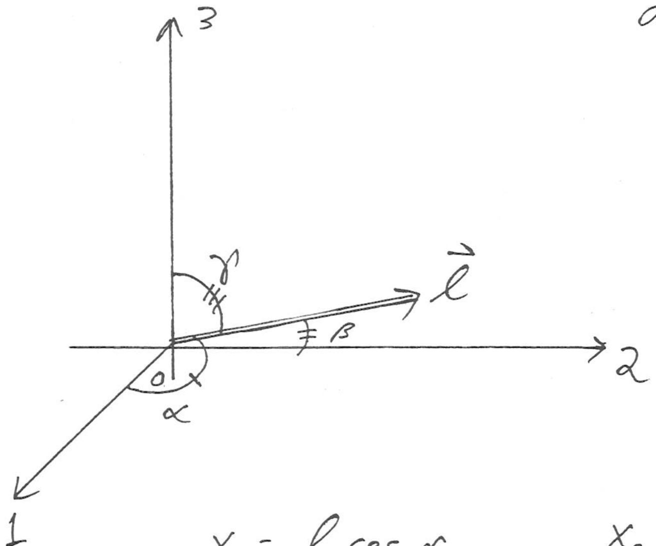
$$\underline{\underline{\lambda}} \, \underline{\underline{\lambda}}^T = \underline{\underline{I}} \iff \sum_{j=1}^{3} \lambda_{ij} (\lambda^T)_{jk} = \delta_{ik}$$

The elements of $\underline{\underline{\lambda}}$ are still direction cosines, but now in 3 dimensions:

λ_{11} is the cosine of the angle between $1'$ and 1 axes

λ_{23} " $2'$ and 3

λ_{ij} " i' and j



α is the angle between the vector $\vec{l}$ and the 1 axis

β between $\vec{l}$ and 2 axis

γ between $\vec{l}$ and 3 axis

$$X_1 = l \cos\alpha \qquad X_2 = l\cos\beta \qquad X_3 = l\cos\gamma$$

The three direction cosines are not independent:

$$(\text{Length of } \vec{l})^2 = l^2 = X_1^2 + X_2^2 + X_3^2 \quad \left(\begin{array}{c}\text{3-dim}\\ \text{Pythagorean}\\ \text{theorem}\end{array}\right)$$

$$l^2 = l^2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)$$

$$\boxed{1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

What does this have to do with rotations in 3-dim?

Take $\vec{l}$ in the direction of the $1'$ axis:

$$\lambda_{11} = \cos\alpha \qquad \lambda_{12} = \cos\beta \qquad \lambda_{13} = \cos\gamma$$

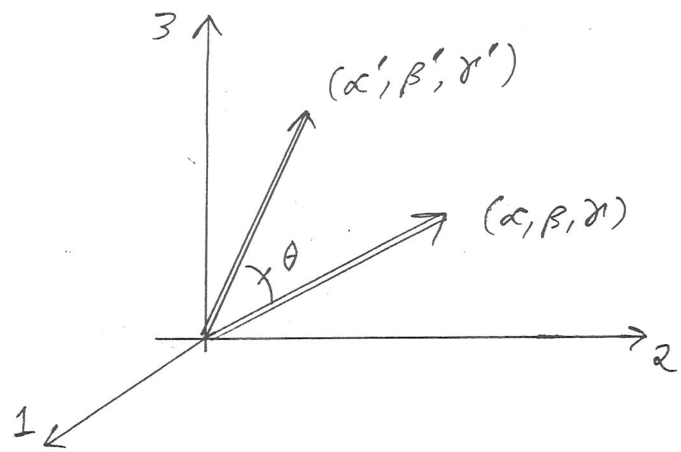
$$\boxed{\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1}$$

Take $\vec{l}$ in the direction of the $2'$ axis:

$$\boxed{\lambda_{21}^2 + \lambda_{22}^2 + \lambda_{23}^2 = 1}$$

$\vec{l}$ along $3'$ $\Longrightarrow$ $\boxed{\lambda_{31}^2 + \lambda_{32}^2 + \lambda_{33}^2 = 1}$

So far, we have 3 constraints on the 9 elements of $\underline{\underline{\lambda}}$. We will get 3 more after a digression into math.



Here are 2 lines with different direction cosines. The angle between the lines is θ.

Part of your homework is to prove that:

$$\cos\theta = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$$

The angle θ between the $1'$ and $2'$ axes is $\frac{\pi}{2}$, so

$$\boxed{\cos\left(\tfrac{\pi}{2}\right) = 0 = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23}}$$

the $2'$ and $3'$ axes are also $\frac{\pi}{2}$ apart;

$$\boxed{0 = \lambda_{21}\lambda_{31} + \lambda_{22}\lambda_{32} + \lambda_{23}\lambda_{33}}$$

And finally, the $3'$ and $1'$ axes are $\frac{\pi}{2}$ apart:

$$\boxed{0 = \lambda_{31}\lambda_{11} + \lambda_{32}\lambda_{12} + \lambda_{33}\lambda_{13}}$$

These equations are 3 more constraints on λ_{ij}. There are a total of 6 constraints on the 9 elements, leaving 3 degrees of freedom for 3-dimensional rotations.

The 6 constraints can be summarised in index notation:

$$\left.\begin{aligned}
\lambda_{11}\lambda_{11} + \lambda_{12}\lambda_{12} + \lambda_{13}\lambda_{13} &= 1\\
\lambda_{21}\lambda_{21} + \lambda_{22}\lambda_{22} + \lambda_{23}\lambda_{23} &= 1\\
\lambda_{31}\lambda_{31} + \lambda_{32}\lambda_{32} + \lambda_{33}\lambda_{33} &= 1\\
\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} &= 0\\
\lambda_{21}\lambda_{31} + \lambda_{22}\lambda_{32} + \lambda_{23}\lambda_{33} &= 0\\
\lambda_{31}\lambda_{11} + \lambda_{32}\lambda_{12} + \lambda_{33}\lambda_{13} &= 0
\end{aligned}\right\} \quad \sum_{j=1}^{3} \lambda_{ij}\lambda_{kj} = \delta_{ik}$$

check this!

And notice that it is exactly the orthogonality relation $\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{I}}$

The 3 degrees of freedom can be seen as follows.

You must specify a direction for the axis of rotation. This requires 2 numbers (latitude and longitude) or (θ and φ in spherical polar coordinates).

Then you must specify the angle of twist about that axis — one number.

Lecture #5 27 Jan 2000

Now that we know how the displacement vector $\vec{x}$ transforms under rotations, we know how <u>any</u> vector transforms

$$\vec{X}' = \underline{\underline{\lambda}}\,\vec{x} \qquad x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j$$

$$\vec{p}' = \underline{\underline{\lambda}}\,\vec{p} \qquad p_i' = \sum_{k=1}^{3} \lambda_{ik} p_k$$

Remember that scalars do not transform at all

$$s' = s$$

and rank-n tensors transform like the exterior product of n displacement vectors. So how does the exterior product $r_i r_j$ transform?

$$r_i' r_j' = \left(\sum_{k=1}^{3} \lambda_{ik} r_k\right)\left(\sum_{l=1}^{3} \lambda_{jl} r_l\right)$$

$$= \sum_k \sum_l \lambda_{ik} \lambda_{jl} (r_k r_l)$$

one factor of λ for each index

$$T_{ij}' = \sum_{k,l} \lambda_{ik} \lambda_{jl} T_{kl}$$

$\uparrow$

2 sums!

The generalization is straight forward

$$T'_{ijk} = \sum_a \sum_b \sum_c \lambda_{ia} \lambda_{jb} \lambda_{kc} T_{abc}$$ (3 indices $\Rightarrow$ 3 λ's)

---

the Dot Product (or Scalar Product)

between two vectors

def $\vec{A} \cdot \vec{B} = \sum_i A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$

The scalar product of two vectors is a scalar. Proof:

$$\vec{A}' \cdot \vec{B}' = \sum_i A'_i B'_i = \sum_i \left(\sum_j \lambda_{ij} A_j \right) \left(\sum_k \lambda_{ik} B_k \right)$$

$$= \sum_j \sum_k \left(\sum_i \lambda_{ij} \lambda_{ik} \right) A_j B_k = \sum_j \sum_k \delta_{jk} A_j B_k$$

$$= \sum_j A_j B_j = \vec{A} \cdot \vec{B}$$

So the scalar product in the primed system is the same as the scalar product in the unprimed system; there is no λ transformation required.

In case you are wondering about the Kronecker delta:

$$\sum_{n=1}^{3} \delta_{kn} X_n = X_k$$

Special case - $k=2$

$$\underbrace{\delta_{21}}_{0} X_1 + \delta_{22} X_2 + \underbrace{\delta_{23}}_{0} X_3 = X_2$$

The Cross Product (or Vector Product)

$$\vec{D} = \vec{A} \times \vec{B}$$

$$D_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_j B_k$$

where ϵ_{ijk} is the Levi-Civita symbol (permutation symbol)

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any indices are the same} \quad \epsilon_{112} = 0,\ \epsilon_{333} = 0 \\ +1, & \text{if } ijk \text{ forms an } \underline{\text{even}} \text{ permutation of } 123 \quad 123,\ 231,\ 312 \\ -1, & \text{if } ijk \text{ forms an } \underline{\text{odd}} \text{ permutation of } 123 \quad 321,\ 213,\ 132 \end{cases}$$

For example:

$$D_1 = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 \quad \text{and the other seven terms} = 0$$

$$= A_2 B_3 - A_3 B_2$$

Triple Product

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{D} \cdot \vec{C} = \sum_i D_i C_i = \sum_i \left(\sum_j \sum_k \epsilon_{ijk} A_j B_k \right) C_i$$

All the indices are dummies – all are summed over so they can be renamed

$i \to K$

$j \to I$

$k \to J$

$$= \sum_K \sum_I \sum_J \epsilon_{KIJ} A_I B_J C_K$$

$$= \sum_K \sum_I \sum_J \epsilon_{IJK} B_J C_K A_I$$

$$= (\vec{B} \times \vec{C}) \cdot \vec{A}$$

$$\boxed{\begin{aligned} \epsilon_{KIJ} &= -\epsilon_{IKJ} \\ &= +\epsilon_{IJK} \end{aligned}}$$

every time you swap two adjacent indices you incur a minus sign.

This is not obvious! $(\vec{A} \times \vec{B})$ and $(\vec{B} \times \vec{C})$ point in completely different directions and have completely different lengths.

The triple product is often written $(\vec{A}, \vec{B}, \vec{C})$ because

$$\vec{A}\times\vec{B}\cdot\vec{C} = \vec{B}\times\vec{C}\cdot\vec{A} = \vec{C}\times\vec{A}\cdot\vec{B}$$
$$= \vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times\vec{B}$$

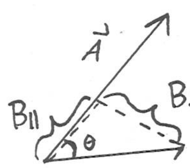
$\boxed{\text{As long as } \vec{A}, \vec{B}, \vec{C} \text{ are in cyclic order, the dot and cross can be anywhere.}}$

Physical Interpretations:

Dot Product:

$\vec{A}\cdot\vec{B} =$ (Length of $\vec{A}$)(the part of $\vec{B}$ that lies along $\vec{A}$) $= |\vec{A}|\, B_{\parallel}$

$=$ (Length of $\vec{B}$)(the part of $\vec{A}$ that lies along $\vec{B}$) $= |\vec{B}|\, A_{\parallel}$



The dot product picks out parallel components of vectors.

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

Cross Product:

$|\vec{A}\times\vec{B}| =$ (Length of $\vec{A}$)(the part of $\vec{B}$ that lies perpendicular to $\vec{A}$)

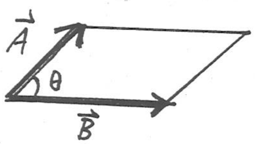
$= |\vec{A}|\, B_{\perp}$

$=$ (Length of $\vec{B}$)(the part of $\vec{A}$ that lies perpendicular to $\vec{B}$)

$= |\vec{B}|\, A_{\perp}$

The cross product picks out perpendicular components of vectors.

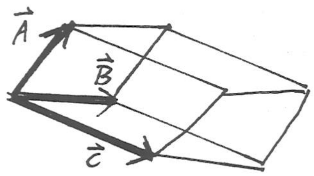
$|\vec{A}\times\vec{B}|$ is the area of the parallelogram with sides along $\vec{A}$ and $\vec{B}$



$$|\vec{A}\times\vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

Triple Product:

$\vec{A} \cdot \vec{B} \times \vec{C}$ is $\pm$ Volume of the parallelopiped with edges along $\vec{A}$, $\vec{B}$, and $\vec{C}$. If $\{\vec{A}, \vec{B}, \vec{C}\}$ form a right-handed system, then $\vec{A} \cdot \vec{B} \times \vec{C}$ is positive.



A useful vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \qquad \text{[BAC-CAB rule]}$$

for the proof (not given here), you need the Levi-Civita identity:

$$\sum_{k=1}^{3} \epsilon_{ijk}\, \epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Suppose that $\hat{e}$ is a unit vector (length 1 unit) in some direction

$$\hat{e} = \frac{\vec{A}}{|\vec{A}|} \qquad \hat{e} \cdot \hat{e} = 1$$

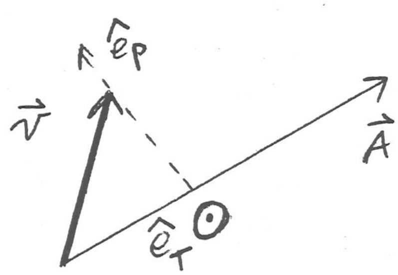
then any vector $\vec{v}$ can be decomposed as

$$\vec{v} = \underbrace{\hat{e}(\vec{v} \cdot \hat{e})}_{\text{component of } \vec{v} \text{ along } \hat{e}} + \underbrace{\hat{e} \times (\vec{v} \times \hat{e})}_{\text{component of } \vec{v} \text{ perpendicular to } \hat{e}}$$

Proof: $\hat{e} \times (\vec{v} \times \hat{e}) = \vec{v}\underbrace{(\hat{e} \cdot \hat{e})}_{1} - \hat{e}(\hat{e} \cdot \vec{v})$ [BAC-CAB rule]

$$\vec{v} = \hat{e}(\vec{v} \cdot \hat{e}) + \hat{e} \times (\vec{v} \times \hat{e}) \quad \blacksquare$$

3-dimensional coordinate-free rotation



rotate $\vec{v}$ by angle φ around axis along $\hat{A}$ (unit vector)

Choose one axis along $\hat{A} = \hat{e}_A$

Choose the second axis (P) through the tip of $\vec{v}$, $\hat{e}_P$

The third axis is $\hat{e}_T = \hat{e}_A \times \hat{e}_P$

The component of $\vec{v}$ along $\hat{A}$ is $v_A = \vec{v} \cdot \hat{A}$

The component of $\vec{v}$ in the P direction is

$$v_P = |\vec{v} - \hat{A}(\vec{v} \cdot \hat{A})| = |\hat{A} \times (\vec{v} \times \hat{A})|$$

The component of $\vec{v}$ in the T direction is $v_T = 0$.

Under the rotation by φ

$v_A' = v_A$ no change to component along axis

$v_P' = v_P \cos\varphi$

$v_T' = v_P \sin\varphi$

$$\vec{v}' = v_A' \hat{e}_A + v_P' \hat{e}_P + v_T' \hat{e}_T$$

$$= v_A \hat{e}_A + v_P \cos\varphi \, \hat{e}_P + v_P \sin\varphi \, \hat{e}_T$$

$$= (\vec{v} \cdot \hat{A})\hat{A} + |\vec{v} - \hat{A}(\vec{v} \cdot \hat{A})| \cos\varphi \, \hat{e}_P + |\vec{v} - \hat{A}(\vec{v} \cdot \hat{A})| \sin\varphi \, \hat{e}_T$$

We are trying to express $\vec{v}'$ in terms of the givens $\hat{A}$, $\vec{v}$, and φ only.

$$\vec{v}' = (\vec{v}\cdot\hat{A})\hat{A} + \underbrace{[\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})]}_{\text{already in the } p \text{ direction}}\cos\varphi + |\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})|\,\hat{e}_T\sin\varphi$$

The last change to make is to write $\hat{e}_T$ as

$$\hat{e}_T = \hat{e}_A \times \hat{e}_p = \hat{A} \times \frac{[\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})]}{|\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})|}$$

and notice that $\hat{A}\times\hat{A} = 0$ in the numerator

$$\hat{e}_T = \frac{\hat{A}\times\vec{v}}{|\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})|}$$

$$\vec{v}' = (\vec{v}\cdot\hat{A})\hat{A} + [\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})]\cos\varphi + \hat{A}\times\vec{v}\sin\varphi$$

$$= \vec{v}\cos\varphi + (\hat{A}\cdot\vec{v})\hat{A}(1-\cos\varphi) + \hat{A}\times\vec{v}\sin\varphi$$

$$v_1' = v_1\cos\varphi + (A_1v_1 + A_2v_2 + A_3v_3)A_1(1-\cos\varphi) + (A_2v_3 - A_3v_2)\sin\varphi$$

$$v_2' = v_2\cos\varphi + (A_1v_1 + A_2v_2 + A_3v_3)A_2(1-\cos\varphi) + (A_3v_1 - A_1v_3)\sin\varphi$$

$$v_3' = v_3\cos\varphi + (A_1v_1 + A_2v_2 + A_3v_3)A_3(1-\cos\varphi) + (A_1v_2 - A_2v_1)\sin\varphi$$

$\vec{v}' = \underline{\underline{\lambda}}\,\vec{v}$ to find λ_{21} for example look in the $2'$ line and write all occurances of v_1

$$\lambda_{21} = A_1A_2(1-\cos\varphi) + A_3\sin\varphi$$

To find λ_{12} look in the $1'$ line and write all occurances of v_2

$$\lambda_{12} = A_2A_1(1-\cos\varphi) - A_3\sin\varphi$$

Here is a complete list of the λ_{ij} direction cosines:

$$\lambda_{11} = \cos\varphi + A_1^2 (1-\cos\varphi)$$

$$\lambda_{22} = \cos\varphi + A_2^2 (1-\cos\varphi)$$

$$\lambda_{33} = \cos\varphi + A_3^2 (1-\cos\varphi)$$

$$\lambda_{12} = A_2 A_1 (1-\cos\varphi) - A_3 \sin\varphi$$

$$\lambda_{21} = A_1 A_2 (1-\cos\varphi) + A_3 \sin\varphi$$

$$\lambda_{23} = A_3 A_2 (1-\cos\varphi) - A_1 \sin\varphi$$

$$\lambda_{32} = A_2 A_3 (1-\cos\varphi) + A_1 \sin\varphi$$

$$\lambda_{31} = A_1 A_3 (1-\cos\varphi) - A_2 \sin\varphi$$

$$\lambda_{13} = A_3 A_1 (1-\cos\varphi) + A_2 \sin\varphi$$

How to use:

1. Decompose the axis $\hat{A}$ into x, y, z components and make sure $\hat{A}$ is a unit vector ($\hat{A} \cdot \hat{A} = 1$). If not, divide $\vec{A}$ by its length $|\vec{A}|$.

2. Once you know A_1, A_2, A_3, and φ you can find the rotation matrix $\underline{\underline{\lambda}}$

3. $\underline{\underline{\lambda}}$ transforms vectors

$$\vec{X}' = \underline{\underline{\lambda}} \, \vec{X}$$

$$\vec{v}' = \underline{\underline{\lambda}} \, \vec{v}$$