$$\frac{F''(\varphi)}{F(\varphi)} = -m^2 \Longrightarrow \frac{d^2F(\varphi)}{d\varphi^2} + m^2F(\varphi) = 0$$

$$F(\varphi) = e^{im\varphi} \longleftarrow Ce^{im\varphi} + De^{-im\varphi}$$

m can be positive or negative.
$m=0$ as well

$\underline{m=0}$ $\quad F(\varphi) = e^{i0\varphi} = 1$

$$\frac{d^2}{d\varphi}(1) + 0^2 1 = 0 \quad \checkmark$$

For problems (like H atom) where the Full range $0 \le \varphi \le 2\pi$ is available.

$$F(\varphi+2\pi) = F(\varphi) \Rightarrow e^{im(\varphi+2\pi)} = e^{im\varphi}$$

$$\Rightarrow e^{im\varphi}\underbrace{e^{im2\pi}}_{1} = e^{im\varphi} \Rightarrow e^{im2\pi} = 1$$

$\Rightarrow m \in$ Integers $\quad$ 1st lie

In Quantum Mechanics, $\Psi(r,\theta,\varphi,t)$ is not observable. $P = |\Psi|^2$

θ equation

$$\sin\theta \frac{d}{d\theta}\left[\sin\theta \frac{dT(\theta)}{d\theta}\right] + \left[\ell(\ell+1)\sin^2\theta - m^2\right]T(\theta) = 0$$

Associated Legendre Differential Equation

solutions: $T_\ell^m(\theta) = T_{\ell m}(\theta) = A\, P_\ell^m(\cos\theta) + B\, Q_\ell^m(\cos\theta)$

$P_\ell^m(\cos\theta)$ = associated Legendre function of the first kind

$Q_\ell^m(\cos\theta)$ = " " " second" kind

If ℓ is a non-negative integer $\ell = 0, 1, 2, 3 \ldots$

and is an integer $-\ell \le m \le +\ell$

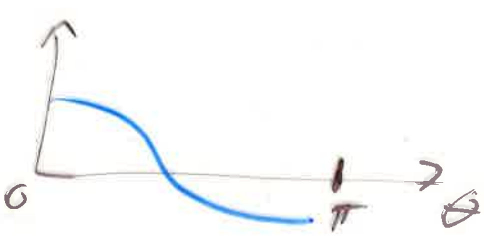
$\rightarrow P_\ell^m(\cos\theta)$ is a polynomial

2nd lie Only need $P_\ell^m(\cos\theta)$.

$Q_\ell^m(\cos\theta)$ are not allowed.

$Q_\ell^m(\cos\theta)$ blows up logarithmically at $\theta = 0, \pi$

$P_1^0(\cos\theta)$



$Q_1^0(\cos\theta)$

In electrostatics — get rid of the $Q_\ell^m(\cos\theta)$ because the voltage (potential) can't be infinite at the poles ($\theta = 0$, $\theta = \pi$)!

In Quantum Mechanics $T_\ell^m(\theta) = A P_\ell^m + B Q_\ell^m$ is part of the wavefunction $\Psi(r,\theta,\phi,t)$

$$\langle \Psi | \Psi \rangle = \text{Probability}$$

$$\langle \psi | \hat{O} | \psi \rangle = \text{expectation value of operator } \hat{O}.$$

$\ell = 0, 1, 2, 3, \ldots$ (orbital angular momentum)

$m = -\ell, -\ell+1, \ldots -1, 0, 1, \ldots \ell-1, \ell.$

ℓ = azimuthal quantum number

m = magnetic quantum number

(n = principal quantum number.)

For Hydrogen, energy levels (bound) depend on n and ℓ, but not m: $E_{n\ell}$

$\ell = 0, m = 0$

$P_0^0(\cos\theta) = 1$

$d = 1$

$\ell = 1 \quad m = +1 : P_1^1(\cos\theta) = \sin\theta$

$m = 0 : P_1^0(\cos\theta) = \cos\theta$

$m = -1 : P_1^{-1}(\cos\theta) = \sin\theta$ ← sign convention

$d = 3 \quad P_\ell^m(\cos\theta) = + P_\ell^{-m}(\cos\theta)$

$\ell = 2 \quad m = +2$

$m = +1$

$m = 0$

$m = -1$

$m = -2$

degeneracy = 5

degeneracy = $(2\ell+1)$

of m's

$Y(\theta, \varphi)$ is also labeled by ℓ, m

If ℓ, m are integral.

$Y_\ell^m(\theta,\varphi) = Y_{\ell m}(\theta,\varphi)$ spherical harmonics

$$Y_\ell^m(\theta,\varphi) \propto T_\ell^m(\theta) F_m(\varphi) = \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

$(-1)^m$ if $m \geq 0$

1 if $m \leq 0$

Just look them up!

Normalized

$$\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[Y_\ell^m(\cos\theta)\right]^* Y_{\ell'}^{m'}(\theta,\varphi) \sin\theta \, d\theta\, d\varphi \cdots = \delta_{\ell\ell'}\delta_{mm'}$$

Remember $dV = r^2 \sin\theta \, dr\, d\theta\, d\varphi$

Radial Eq.

$$\frac{d}{dr}\left[r^2 \frac{dR}{dr}\right] - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R(r) = \ell(\ell+1)\,R(r)$$

$$R(r) = \frac{u(r)}{r}$$

$$\Rightarrow -\frac{\hbar^2}{2m} u''(r) + \left[V(r) + \underbrace{\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}}_{\text{centrifugal term}}\right]u(r) = E\,u(r)$$