Any operator $\hat{Q}$ is Hermitian if: $\int \Psi^*(\vec{x},t)\, \hat{Q}\, \Psi(\vec{x},t)\, d^3x$

$$= \int \left(\hat{Q}\,\Psi(\vec{x},t)\right)^* \Psi(\vec{x},t)\, d^3x$$

$$= \langle \hat{Q} \rangle$$

$$= \langle \hat{Q} \rangle^* \qquad \text{"Real"}$$

Properties of Hermitian Operators

In general for any function Φ, $\hat{Q}\Phi = \Phi'$

e.g. $\hat{P}\Phi = \Phi' = \nabla \Phi$

but if it happens that $\hat{Q}\Phi_a = q_a \Phi_a$ "Eigenvalue Equ"

$\hookrightarrow$ # "eigenvalue"

then Φ is said to be an eigenfunction of $\hat{Q}$ with eigenvalue q.

Claim: 1) Eigenvalues of Hermitian Operators are Real.

2) Eigenfunctions of Hermitian Operators corresponding to distinct eigenvalues are orthogonal.

In Linear Algebra: (Aside)

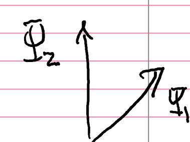
$$\underset{\sim}{M}\,\underset{\sim}{V} = a\underset{\sim}{V} \quad = \quad (\underset{\sim}{M} - a)\underset{\sim}{V} = 0$$

for 2x2:

$$\begin{pmatrix} m_{11} - a & m_{12} \\ m_{21} & m_{22} - a \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

for non-trivial Soln we require that

$$\det \begin{pmatrix} m_{11} - a & m_{12} \\ m_{21} & m_{22} - a \end{pmatrix} = 0$$



$$(m_{11} - a)(m_{22} - a) - m_{12} m_{21} = 0$$

Inner Product of Two Wave functions:

$$\langle \Psi_1(\vec{x},t) | \Psi_2(\vec{x},t) \rangle = \int \Psi_1^*(\vec{x},t)\, \Psi_2(\vec{x},t)\, d^3x \neq \langle \Psi_2(\vec{x},t) | \Psi_1(\vec{x},t) \rangle$$

"Inner Product" "Complex"

$$\vec{V} \cdot \vec{W} = V_1 W_1 + V_2 W_2 + V_3 W_3 = \sum^{\infty} V_i W_i \rightarrow \int V(x)\, W(x)\, dx$$

"Inner Product" "Complex"

$$\vec{V}\cdot\vec{W} = V_1W_1 + V_2W_2 + V_3W_3 = \sum_{i=1}^{\infty} V_iW_i \longrightarrow \int V(x)\,W(x)\,dx$$

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Proof of statement (1)

Lets compute the Expectation Value of $\hat{Q}$ using its own eigenfunctions:

$$\langle\hat{Q}\rangle = \int d^3x\left(\Phi_a^*(\vec{x},t)\,\hat{Q}\,\Phi_a(\vec{x},t)\right)$$

$$= q_a \int d^3x\ \Phi_a^*(\vec{x},t)\,\Phi_a(\vec{x},t)$$

$$= \langle\hat{Q}\rangle^* \qquad \text{"because } \hat{Q} \text{ is Hermitian"}$$

$$\Rightarrow q_a = q_a^* \qquad \text{Q.E.D.}$$

Proof of statement (2):

(i) $\hat{Q}\,\Phi_a = q_a\,\Phi_a$; (ii) $\hat{Q}\,\Phi_b = q_b\,\Phi_b$ $(q_a \neq q_b)$

(i) $\int \Phi_b^*\,\hat{Q}\,\Phi_a\,d^3x = q_a \int \Phi_b^*\,\Phi_a\,d^3x$

(ii) $\int \Phi_a^*\,\hat{Q}\,\Phi_b\,d^3x = q_b \int \Phi_a^*\,\Phi_b\,d^3x$

$\int (\hat{Q}\,\Phi_a)^*\,\Phi_b\,d^3x = q_b \int \Phi_a^*\,\Phi_b\,d^3x$ by Hermiticity of $\hat{Q}$

(ii) $\int \Phi_b^*\,\hat{Q}\,\Phi_a\,d^3x = q_b^* \int \Phi_b^*\,\Phi_a\,d^3x$ "take complex conjugate"

$$0 = (q_a - q_b^*)\int \Phi_b^*\,\Phi_a\,d^3x$$

$$\Rightarrow (q_a - q_b)\int \Phi_b^*\,\Phi_a\,d^3x = 0$$

$$\int \Phi_b^*(\vec{x},t)\,\Phi_a(\vec{x},t)\,d^3x = 0 \quad \text{for } q_a \neq q_b$$

Q.E.D.

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General Properties of Solutions of the TDSE:

$$\hat{H}\,\Psi(\bar{x},t) = i\hbar \frac{\partial}{\partial t}\Psi(\bar{x},t)$$

$$\hat{H} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x})$$

↳ time independent

$$\hat{H} \neq \hat{H}(t)$$

Formal Particular Solution

$$\Psi(\vec{x},t) = e^{-\frac{i}{\hbar}\hat{H}t}\,\psi(\vec{x})$$

Use: $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}\hat{H}\,e^{-i/\hbar \hat{H}t}\,\psi(\vec{x})$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\,\Psi$$

Since $\hat{H}$ is a Hermitian Operator $\hat{H}\psi_E(\vec{x}) = E\,\psi_E(\vec{x})$

Since the TDSE & TISE are linear in Ψ or ψ respectively it means they obey a superposition principle:

i.e. $\Psi_1(\bar{x},t)$ & $\Psi_2(\bar{x},t)$ are solutions

then $\Psi(x,t) = \left(a\Psi_1(\bar{x},t) + b\,\Psi_2(\bar{x},t)\right)$ is also a solution.

We can $\psi(\vec{x}) = \sum_E a_E\,\psi_E(\vec{x})$

$$\Psi(\vec{x},t) = e^{-\frac{i}{\hbar}\hat{H}t}\sum_E a_E\,\psi_E(\vec{x})$$

$$\Psi(\vec{x},t) = \sum_E a_E\,e^{-\frac{i}{\hbar}Et}\,\psi_E(\vec{x})$$

$\hat{H}\psi_E(\vec{x}) = E\psi_E(\vec{x})$

this a test

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