Class 6 - Commutability, Compatibility and the Uncertainty Relation

Thursday, September 6, 2018 7:00 AM

$$\hat{H}\,\Psi(\vec{x},t) = i\hbar \frac{\partial}{\partial t}\Psi(\vec{x},t) \qquad \text{"TDSE"}$$

If $\int d^3x\, \Psi^*(\vec{x},t)\,\Psi(\vec{x},t) = 1$

$$\boxed{\Psi(\vec{x},t) = \sum_n c_n e^{-i/\hbar E_n t}\,\psi_n(x)}$$

$$\Rightarrow \underbrace{\sum_n |c_n|^2 = 1}$$ "Completeness Relation"

only true if $\hat{H} \neq \hat{H}(t)$

$$\hat{H}\,\psi_n(\vec{x}) = E_n\,\psi_n(\vec{x}) \qquad \text{"TISE"}$$

$$\langle \hat{H} \rangle = \int d^3\vec{x}\, \Psi(\vec{x},t)\,\hat{H}\,\Psi(\vec{x},t) = \sum_n |c_n|^2 E_n$$ "Conservation of Energy"

$|c_n|^2$: prob. that if we measure the energy on a system represented by $\Psi(\vec{x},t)$ we will obtain the value E_n

The possible result of measuring the energy is restricted to the set of values $\{E_1, E_2, E_3, \ldots, E_N\}$

The set of values c_n is determined by the initial conditions of the system.

$$c_n = \int d^3x'\, \psi_n^*(\vec{x}')\,\Psi(\vec{x}',0)$$

In Linear Algebra, in a basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$$\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 = V_i\,\hat{e}_i$$

$$V_i = \hat{e}_i \cdot \vec{V} \}$$

$$\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix};\ \hat{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix};\ \hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\vec{V} = \underbrace{(\hat{e}_i\,\hat{e}_i)}\cdot\vec{V}$$

$$(1\ 0\ 0) \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} =$$

(outer product)

$$\hat{e}_1\hat{e}_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} \qquad \hat{e}_3\hat{e}_3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$\hat{e}_2\hat{e}_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$\hat{e}_i\,\hat{e}_i = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \mathbb{1} \qquad \sum_i |\hat{e}_i|^2 = \mathbb{1} \quad *$$

$$\sum_i |V_i|^2 = |\vec{V}|^2 = 1$$

in component form:

$$* \left(\sum_i |\hat{e}_i|^2\right)_{\ell m} = \delta_{\ell m} \quad *$$

"Closure Relation"

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Closure Relation In Q.M.

$$\vec{A}\cdot\vec{B} = A_i B_i$$

$$\vec{A}\cdot\vec{B}\ \vec{C}\cdot\vec{D} = A_i B_i C_j D_j$$

$$\Psi(\vec{x},t)\Big|_{t=0} = \sum_n c_n e^{-i/\hbar E_n t}\,\psi_n(\vec{x})\Big|_{t=0}$$

$$\underbrace{\Psi(\vec{x},0)}_{*} = \sum_n \underbrace{\int d^3x'\,\psi_n^*(\vec{x}')\,\Psi(\vec{x}',0)}\,\psi_n(\vec{x})$$

$$\underbrace{\Psi(\vec{x},0)}_{*} = \sum_n \int d^3x'\, \psi_n^*(\vec{x}')\, \Psi(\vec{x}',0)\, \psi_n(\vec{x})$$

$$= \int d^3x' \left(\sum_n \psi_n^*(\vec{x}')\, \psi_n(\vec{x}) \right) \Psi(\vec{x}',0)$$

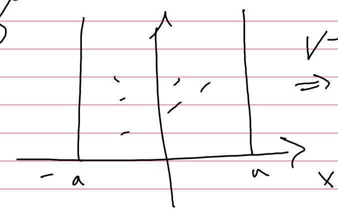
$$\Longrightarrow \sum_n \psi_n^*(\vec{x}')\, \psi_n(\vec{x}) = \delta^3(\vec{x}-\vec{x}') \quad *$$

"Closure Relation"

$$\delta^3(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$(\hat{e}_i)_l \qquad \psi_n(\vec{x}) \qquad \vec{x} = (x,y,z)$

For Example for a Particle in a Box



$V \to \infty \Rightarrow \psi = 0$

$$V(x) = \begin{cases} 0 & |x| < a \\ \infty & \forall \text{ all other } x \end{cases}$$

In the Region $x : \{-a, a\}$ $\quad V(x) = 0 \Rightarrow H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$$\Longrightarrow \text{In } \{-a,a\} \Rightarrow H\psi(x) = E\psi(x)$$

$$\psi(x) = A e^{\pm ikx} \; ; \quad E = \frac{\hbar^2 k^2}{2m}$$

B.C. $\quad \psi(a) = \psi(-a) = 0$

$\Longrightarrow$ k is quantized $\quad k_n = \frac{n\pi}{a} \} \quad n = 1, 2, \ldots$

$$\psi(x) = \frac{e^{ikx} \pm e^{-ikx}}{\sqrt{2}} \qquad \cos kx, \; \sin kx$$

Normalization: $\int_{-a}^{a} \psi^*(x)\, \psi(x)\, dx = 1 \Rightarrow A = \sqrt{\frac{2}{a}}$

$$\psi_n^{(e)}(x) = \sqrt{\frac{2}{a}} \cos k_n x \quad n: \text{odd} \qquad \psi^{(o)}(x) = \sqrt{\frac{2}{a}} \sin k_n x \quad n: \text{even}$$

$$\left(k_n = \frac{n\pi}{2a} \right)$$

$$\frac{2}{a} \sum_n \cos k_n x \cos k_n x' = \delta(x-x')$$

Commutability, compatibility & the Uncertainty Relations

Recall, $\hat{P}_l = -i\hbar \frac{\partial}{\partial x_l}$ "operator"

$\hat{X}_l = X_l$

Defn: commutator of two operators $\hat{A}$ & $\hat{B}$: $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

$$[\hat{X}_l, \hat{P}_m]\, \Psi(\vec{x},t) = (\hat{X}_l \hat{P}_m - \hat{P}_m \hat{X}_l)\, \Psi(\vec{x},t)$$

$$= -i\hbar \left(X_l \frac{\partial}{\partial X_m} - \frac{\partial}{\partial X_m} X_l \right) \Psi(\vec{x},t)$$

$$[X_l, P_m]\Psi(x,t) = (X_l P_m - P_m X_l)\Psi(x,t)$$

$$= -i\hbar\left(X_l \frac{\partial}{\partial X_m} - \frac{\partial}{\partial X_m} X_l\right)\Psi(\bar{x},t)$$

$$= +i\hbar\left(\frac{\partial}{\partial X_m} X_l\right)\Psi(\bar{x},t) = +i\hbar\,\delta_{ml}$$

$$[\hat{X}_l, \hat{P}_m] = i\hbar\,\delta_{ml}$$

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Thm: If two operators commute then we can find (& are Hermitian) simultaneous eigenfunctions.

In general, $\hat{A}\Psi(\bar{x},t) = \Psi'(\bar{x},t)$

$$\hat{B}\left(\hat{A}\phi(\vec{x}) = a\,\phi(\vec{x})\right)$$

$$\hat{A}\left(\hat{B}\phi(\vec{x}) = b\,\phi(\vec{x})\right)$$

$$\hat{B}\hat{A}\phi(\vec{x}) = a\hat{B}\phi(\vec{x}) = ab\,\phi(\vec{x})$$

$$\left.\begin{aligned}&(\hat{B}\hat{A} - \hat{A}\hat{B})\phi(\vec{x}) = 0\\ &\hat{B}\hat{A} - \hat{A}\hat{B} = 0 \Rightarrow [\hat{A},\hat{B}] = 0\end{aligned}\right\}$$

If $[\hat{A},\hat{B}] = 0$ & $\hat{A}^\dagger = \hat{A}$, $\hat{B}^\dagger = \hat{B}$ $\Longrightarrow (\hat{A}\hat{B})^\dagger = \hat{A}\hat{B}$

$$\hat{A}\hat{B}\,\phi(x) = ab\,\phi(x)$$

The converse is true if $[\hat{A},\hat{B}] = 0 \Rightarrow$ we can find simultaneous eigenfunction

$$[\hat{A},\hat{B}]\psi(x) = 0 \qquad \hat{A}\hat{B}\psi(x) = \hat{B}\hat{A}\psi(x)$$

sps $\hat{B}\psi(x) = b\,\psi(x)$

$$\hat{A}\,b\,\psi(x) = \hat{B}\hat{A}\psi(x)$$

$$\hat{B}\left(\hat{A}\psi(x)\right) = b\left(\hat{A}\psi(x)\right)$$

$\Rightarrow \hat{A}\psi(x)$ is an eigenfunction of $\hat{B}$ w/ eigenvalue b.

$$\Longrightarrow \hat{A}\psi(x) = a\,\psi(x)$$

The expression $\hat{A}\Psi(\vec{x},t)$ ~~is~~ representative of the measurement of $\hat{A}$ in a state by $\Psi(\vec{x},t)$