Recall that if two operators commute, $[\hat{A},\hat{B}]=0$
then we can find simultaneous eigenfunctions

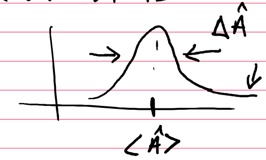
$$\hat{A}\varphi_{ab} = a\varphi_{ab} \quad ; \quad \hat{B}\varphi_{ab} = b\,\varphi_{ab}$$

$$\hat{A}\hat{B}\varphi_{ab} = ab\,\varphi_{ab}$$

$\Rightarrow$ $\hat{A}$ & $\hat{B}$ are compatible.

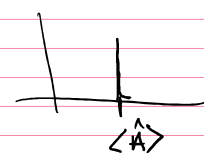
The uncertainty in the measurement of an operator $\hat{A}$ is given by,

$$\Delta\hat{A} \equiv \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$$



$$\langle\hat{A}\rangle = \int \Psi^*(\vec{x},t)\,\hat{A}\,\Psi(\vec{x},t)\,d^3x$$

$$\uparrow$$
$$\varphi_{ab}$$



$A = (q, p)$
$\to \hat{A} = (\hat{q}, \hat{p})$

If $\Psi = \varphi_{ab}$

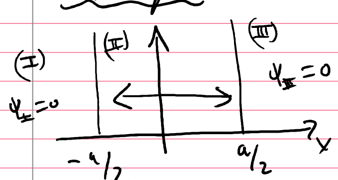
$$\Rightarrow \quad \langle\hat{A}\rangle = \int d^3x\, \varphi_{ab}^*(\vec{x})\,\hat{A}\,\varphi_{ab}(\vec{x})$$
$$= a\int d^3x\, \varphi_{ab}^*(\vec{x})\,\varphi_{ab}(\vec{x}) = a$$

$$\langle\hat{A}^2\rangle = a^2$$

$$\to \quad \Delta\hat{A} = 0 \quad ; \quad \Delta\hat{B} = 0 \; ; \quad \Delta(\hat{A}\hat{B}) = 0$$

$$(\Delta\hat{A})(\Delta\hat{B}) = 0 \qquad \text{when } \Psi = \varphi_{ab}$$

Example:



$$V(x) = \begin{cases} 0 & -\frac{a}{2} < x \le a/2 \\ \infty & \text{otherwise} \end{cases}$$

$$\hat{H}\psi = E\psi \quad ; \quad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x) \qquad \frac{d^2}{dx^2}\psi(x) = -\frac{2mE}{\hbar^2}\,\psi(x)$$

$E > 0$

$$\frac{d^2}{dx^2}\psi(x) = -k^2\psi(x)$$

$\hookrightarrow$ positive definite

$$\psi(x) = \begin{cases} A e^{+ikx} \\ B e^{-ikx} \end{cases} \qquad E = \frac{\hbar^2 k^2}{2m} \geq 0$$

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

$$\psi(x)\Big|_{x=\pm a/2} = 0 \qquad A e^{ika/2} + B e^{-ika/2}$$

$$\psi(x)\Big|_{x=\pm a/2} = 0 \qquad A e^{ik a/2} + B e^{-ik a/2} = 0$$

$$A\left(\cos\frac{ka}{2} + i\sin\frac{ka}{2}\right) + B\left(\cos\frac{ka}{2} - i\sin\frac{ka}{2}\right) = 0$$

$$(A+B)\cos\frac{ka}{2} + (A-B)\sin\frac{ka}{2} = 0$$

$$A = B \implies \cos\frac{ka}{2} = 0 \implies \frac{ka}{2} = \frac{(2n+1)\pi}{2}$$

$$\implies k_n = \frac{(2n+1)\pi}{a}$$

$$A = -B \implies \sin\frac{ka}{2} = 0 \implies \frac{ka}{2} = n\pi$$

$$k_n = \frac{2n\pi}{a}$$

$$\psi_+ = \psi_e = A\cos\frac{k_n a}{2}, \quad n = \text{odd} \qquad k_n = \frac{n\pi}{a}$$

$$\psi_- = \psi_o = A\sin\frac{k_n a}{2}; \quad n = \text{even}$$

Alternately, $\psi_\pm = A\left(e^{ikx} \pm e^{-ikx}\right)$

$$\psi_e(-x) = \psi_e(x) \quad ; \quad \psi_o(-x) = -\psi_o(x)$$

Note: $H = H(x) \qquad H(-x) = H(x)$ "even in x"

Defn Parity Operator $\hat{\mathbb{P}}$:

$\psi(x)$ $\qquad \hat{\mathbb{P}}\,\Psi(\vec{x}) = \Psi(-\vec{x})$

$$x \xrightarrow{\hat{\mathbb{P}}} -x$$
$$y \xrightarrow{\hat{\mathbb{P}}} -y$$
$$z \xrightarrow{\hat{\mathbb{P}}} -z$$

for the Infinite Potential Well $[\hat{\mathbb{P}}, \hat{H}] = 0$

$$\hat{H}\psi(x) = \psi'(x) \implies \hat{\mathbb{P}}\hat{H}\psi(x) = \mathbb{P}\psi'(x) = \underline{\psi'(-x)}$$

$$\hat{\mathbb{P}}\hat{H}\psi(x) = \psi'(-x)$$

$$\hat{H}\hat{\mathbb{P}}\psi(x) = \hat{H}\,\underline{\underline{\psi(-x)}}$$

$$\psi'(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)$$

(I) $\psi'(-x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(-x) + V(-x)\psi(-x)$ ✱

(II) $\hat{H}\psi(-x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(-x) + V(x)\psi(-x)$

$\therefore$ If $V(x) = V(-x) \Rightarrow \hat{H}\psi(-x) = \psi'(-x)$

$$\Rightarrow \hat{H}\hat{\mathbb{P}}\psi(x) - \hat{\mathbb{P}}\hat{H}\psi(x) = 0$$

$$[\hat{H}, \hat{\mathbb{P}}]\psi(x) = 0$$

$$[\hat{H}, \hat{\mathbb{P}}] = 0 \Rightarrow$$ $\hat{H}$ & $\hat{\mathbb{P}}$ can have simultaneous eigenfunctions

Eigenfunctions of $\hat{\mathbb{P}}$

$$\hat{\mathbb{P}}^2\psi(x) = \eta\,\psi(x)$$ phase $|\eta|^2 = 1$

$$\hat{\mathbb{P}}^2 = \mathbb{1}$$

Since $\mathbb{P}$ is Hermitian its eigenvalues must be real

$\Rightarrow$ Eigenvalue of $\hat{\mathbb{P}}^2$ is 1.

$\Rightarrow$ Eigenvalues of $\mathbb{P}$ must be ± 1

$$\mathbb{P}\psi_e(x) = \psi_e(-x) = \underset{\uparrow\atop 1}{}\psi_e(x)$$

$$\mathbb{P}\psi_o(x) = -\psi_o(x)$$

$$\int \Psi^*(x,t)\,\hat{\mathbb{P}}\,\Psi(x,t)\,dx \overset{?}{=} \int \left(\hat{\mathbb{P}}\Psi(x,t)\right)^*\Psi(x,t)\,dx$$

$$\int \Psi^*(x,t)\,\Psi(-x,t)\,dx \overset{?}{=} \underbrace{\int \Psi^*(-x,t)\,\Psi(x,t)\,dx}_{x\to -x \quad dx \to -dx}$$

$$= \int \Psi^*(x,t)\,\Psi(-x,t)\,dx$$

For the infinite potential $Ae^{\pm ikx}$ are eigenfunctions of $\hat{H}$ w/ eigenvalue $E = \frac{\hbar^2 k^2}{2m}$, but these are not eigenfunctions of $\hat{\mathbb{P}}$.

But I can form Linear combinations $\ni$ they are eigenfunctions of both $\hat{H}$ & $\hat{\mathbb{P}}$

$$\psi_\pm(x) = A\left(e^{+ikx} \pm e^{-ikx}\right)$$

$$\hat{\mathbb{P}}\psi_\pm(x) = \pm\,\psi_\pm(x)$$

Note: $\psi_1 = A e^{ikx}$; $\psi_2 = B e^{-ikx}$

$$H\psi_1 = E_1\psi_1 \quad ; \quad H\psi_2 = E_2\psi_2$$

But! $E_1 = E_2 = \frac{\hbar^2 k^2}{2m}$ "degeneracy"

If there $\exists$ an operator $\hat{Q} \ni [\hat{H}, \hat{Q}] = 0$

$\Rightarrow$ degeneracy (in general, but not always)

$$\hat{H}\psi_1 = E\psi_1$$

$$\hat{Q}\psi_1 = \psi_1' \quad ; \quad \hat{H}\hat{Q}\psi_1 = \hat{H}\psi_1'$$

$$\hat{Q}\hat{H}\psi_1 = \hat{Q}E\psi_1 = E\hat{Q}\psi_1 = E\psi_1'$$

$$\hat{H}\hat{Q}\psi_1 = \hat{H}\psi_1'$$

$$\hat{Q}\hat{H}\psi_1 = E\psi_1'$$

$$[\hat{H},\hat{Q}]\psi_1 = \underbrace{(\hat{H}-E)\psi_1'}_{} = 0 \quad \text{if } [\hat{H},\hat{Q}]=0$$

$$\underbrace{\hat{H}\psi_1' = E\psi_1'}_{} \Rightarrow \psi_1' \text{ is an eigenfunction of } \hat{H} \text{ w/ eigenvalue } E$$

∴ Both ψ_1 & ψ_1' are eigenfunctions of $\hat{H}$ w/ the same eigenvalue! $\Rightarrow$ If ψ_1' is distinct from ψ_1 then there is a degeneracy.

$$\psi_1' = \hat{Q}\psi_1$$

If the eigenfunctions of $\hat{Q}$: $\quad \hat{Q}\varphi_q = q\varphi_q$

$$q : \{q_1, q_2, q_3\}$$

$$\hat{H}\psi = E\psi$$

$$[\hat{H},\hat{Q}]=0 \qquad \psi = \psi_{Eq}$$

$$\hat{H}\psi_{Eq} = E\psi_{Eq} \quad ; \quad \hat{Q}\psi_{Eq} = q\psi_{Eq}$$

$$\underbrace{\psi_{Eq_1},\ \psi_{Eq_2}\ \&\ \psi_{Eq_3}}_{\Rightarrow\ 3\text{-fold degeneracy.}} \text{ are distinct}$$

If $\hat{H}$ & $\hat{Q}$ commute $\Rightarrow$ the eigenvalues are E are not suficient to distinguish the different states

We can generalize this to any # of commuting operators.
If the set of operators $\{\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \ldots\}$ all commute amongst each other & w/ $\hat{H}$, then the quantum will only be uniquely characterized by specifying the eigenvalues of $\{\hat{H}, \hat{Q}_1, \hat{Q}_2, \ldots\}$, i.e. we would label the eigenfunctions by,

$$\Psi_{Eq_1q_2q_3\ldots}$$

The eigenvalues of a complete set of operators:

$$\{\hat{H}, \hat{Q}_1, \hat{Q}_2, \ldots\}$$

which are all mutually commuting will uniquely characterize a Quantum State.

the set of eigenvalues $\{E, q_1, q_2, \ldots\}$ are called the quantum Numbers.