If $[\hat{Q}, \hat{H}] = 0 \quad \exists \quad \hat{H}\psi_{Eq} = E\psi_{Eq} \; ; \; \hat{Q}\psi_{Eq} = q\psi_{Eq}$

If $\hat{H}\psi_1 = E_1\psi_1 \Rightarrow \hat{H}(\hat{Q}\psi_1) = E_1(\hat{Q}\psi_1)$

$\Rightarrow \hat{Q}\psi_1 = \psi'$ is an Eigenfunction of $\hat{H}$ w/ the same eigenvalue.

If $\psi' \not\propto \psi_1 \Rightarrow \psi'$ & ψ_1 are two distinct states corresponding to the same energy.

Quantum state is characterized by the eigenvalues corresponding to a complete set of operators

$\{\hat{H}, \hat{Q}_1, \hat{Q}_2, \ldots\} \quad [\hat{Q}_i, \hat{Q}_j] = 0 \; ; \; [\hat{H}, \hat{Q}_i] = 0 \quad \forall i$

The Uncertainty Relation

Thm: If $[\hat{A}, \hat{B}] \neq 0$ & $\hat{A}^\dagger = \hat{A}$, $\hat{B}^\dagger = \hat{B}$

then $\boxed{\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right|}$

where, $\Delta\hat{A} = \left[\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2\right]^{1/2}$

Proof: Defn. $\hat{A}_s \equiv \hat{A} - \langle\hat{A}\rangle\mathbb{1} \; ; \; \hat{B}_s \equiv \hat{B} - \langle\hat{B}\rangle\mathbb{1}$

$\langle\hat{A}_s\rangle = \langle\hat{A}\rangle - \langle\hat{A}\rangle\langle\mathbb{1}\rangle = 0$ ✓

$\langle\hat{B}_s\rangle = 0$

$(\Delta\hat{A}_s)^2 = \langle\hat{A}_s^2\rangle - \cancelto{0}{\langle\hat{A}_s\rangle^2} = \langle\hat{A}_s^2\rangle \qquad \langle\langle\hat{A}\rangle\rangle = \langle\hat{A}\rangle$

$(\Delta\hat{B}_s)^2 = \langle\hat{B}_s^2\rangle$

$(\Delta\hat{A})^2 = (\Delta\hat{A}_s)^2 = \langle\hat{A}_s^2\rangle = \langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle$

$= \langle\hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2\rangle$

$$= \langle \hat{A}^2 \rangle - 2\langle \hat{A} \rangle \langle \hat{A} \rangle + \langle \hat{A} \rangle^2$$

$$= \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \quad \checkmark$$

$$(\Delta \hat{A})^2 (\Delta \hat{B})^2 = (\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2$$

$$= \langle \hat{A}_s^2 \rangle \langle \hat{B}_s^2 \rangle$$

$$\langle \hat{A}_s^2 \rangle = \int \Psi^*(\vec{x},t)\, \hat{A}_s^2\, \Psi(x,t)\, d^3X$$

$$= \int \left(\hat{A}_s \Psi(x,t)\right)^* \left(\hat{A}_s \Psi(x,t)\right) d^3X$$

$$\hat{A}_s \Psi(x,t) = \varphi(\vec{x},t)$$

$$\hat{B}_s \Psi(x,t) = \psi(\vec{x},t)$$

$$= \int \varphi^*(\vec{x},t)\, \varphi(\vec{x},t)\, d^3X = \langle \varphi(\vec{x},t) | \varphi(\vec{x},t) \rangle$$

$$\langle \hat{B}_s^2 \rangle = \langle \psi(\vec{x},t) | \psi(\vec{x},t) \rangle$$

Schwarz Inequality:

$$|\langle \varphi | \psi \rangle|^2 \leq \langle \varphi | \varphi \rangle \langle \psi | \psi \rangle$$

$$\left| \int \varphi^*(\vec{x},t)\, \psi(\vec{x},t)\, d^3x \right|^2 \leq \int \varphi^*(\vec{x},t)\, \varphi(\vec{x},t)\, d^3X \times \int \psi^*(\vec{x},t)\, \psi(\vec{x},t)\, d^3x'$$

$$|\vec{V} \cdot \vec{W}|^2 \leq \vec{V}^2 \vec{W}^2$$

The equality is true if $\vec{V} \propto \vec{W}$

Likewise in our case the equality holds if $\varphi = c\,\psi$.

$$(\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 = \langle \varphi | \varphi \rangle \langle \psi | \psi \rangle \geq |\langle \varphi | \psi \rangle|^2$$

"By Schwarz Inequality"

$$(\Delta \hat{A}_s)(\Delta \hat{B}_s)^2 \geq |\langle \varphi | \psi \rangle|^2$$

$$\geq \Bigg| \underbrace{\int \varphi^*(\vec{x},t)\, \psi(\vec{x},t)\, d^3x}_{\int \left(\hat{A}_s(\vec{x},t)\Psi(\vec{x},t)\right)^* \left(\hat{B}_s \Psi(\vec{x},t)\right) d^3X} \Bigg|^2$$

$$= \int \Psi^*(\vec{x},t)\, \hat{A}_s \hat{B}_s \Psi(\vec{x},t)\, d^3X$$

$$= \int \Psi^*(\vec{x},t)\, \hat{A}_s \hat{B}_s \Psi(\vec{x},t)\, d^3x$$

$$(\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 \geq |\langle \hat{A}_s \hat{B}_s \rangle|^2$$

$$\hat{A}_s \hat{B}_s = \frac{\hat{A}_s \hat{B}_s - \hat{B}_s \hat{A}_s + \hat{B}_s \hat{A}_s + \hat{A}_s \hat{B}_s}{2}$$

$$= \frac{1}{2}[\hat{A}_s, \hat{B}_s] + \frac{1}{2}[\hat{A}_s, \hat{B}_s]_+$$

$$(\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 \geq \frac{1}{4} |\langle [\hat{A}_s, \hat{B}_s] \rangle + \langle [\hat{A}_s, \hat{B}_s]_+ \rangle|^2$$

$$\underbrace{\langle \hat{A}_s \hat{B}_s \rangle}_{\text{complex \#}} = \int \Psi^*(\vec{x},t)\, \hat{A}_s \hat{B}_s \Psi(\vec{x},t)\, d^3x$$

$$\langle \hat{A}_s \hat{B}_s \rangle^* = \left[\int \Big(\underbrace{\hat{A}_s \Psi(\vec{x},t)}\Big)^* \underbrace{\hat{B}_s \Psi(\vec{x},t)}\, d^3x \right]^*$$

$$= \int \left(\hat{B}_s \Psi(\vec{x},t)\right)^* \hat{A}_s \Psi(\vec{x},t)\, d^3x$$

$$= \int \Psi^*(\vec{x},t)\, \hat{B}_s \hat{A}_s \Psi(\vec{x},t)\, d^3x$$

$$\langle \hat{A}_s \hat{B}_s \rangle^* = \langle \hat{B}_s \hat{A}_s \rangle$$

i.e. $\langle \varphi | \psi \rangle^* = \langle \psi | \varphi \rangle$

$$\Rightarrow \langle [\hat{A}_s, \hat{B}_s] \rangle^* = - \langle [\hat{A}_s, \hat{B}_s] \rangle \equiv i a$$ Real "pure imaginary" $\equiv i a$

$$\langle [\hat{A}_s, \hat{B}_s]_+ \rangle^* = \langle [\hat{A}_s, \hat{B}_s]_+ \rangle = b$$ "pure real"

$$(\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 \geq \frac{1}{4} \underbrace{|ia + b|^2}_{= |a|^2 + |b|^2}$$

$$|b + ia|^2 = (b + ia)(b - ia) = b^2 + a^2$$

$$\boxed{(\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 \geq \frac{1}{4}\left[|\langle [\hat{A}_s, \hat{B}_s]_- \rangle|^2 + |\langle [\hat{A}_s, \hat{B}_s]_+ \rangle|^2 \right]}$$

$\phi \propto \psi$

$\hat{A}_s \Psi(x,t) = c\, B_s \Psi(x,t)$

Both terms on the RHS are positive definite,

$$\Rightarrow \quad (\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 \geq \frac{1}{4} |\langle [\hat{A}_s, \hat{B}_s]_- \rangle|^2$$

$$(\Delta \hat{A})^2 (\Delta \hat{B})^2 \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2$$

$$\ast \quad (\Delta \hat{A})(\Delta \hat{B}) \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| \quad \Rightarrow \quad (\Delta \hat{A} \Delta \hat{B}) \geq \frac{1}{2} \langle [\hat{A}, \hat{B}]_+ \rangle$$

Q.E.D.

why not use: $(\Delta \hat{A}_s)^2 (\Delta \hat{B}_s)^2 \geq \frac{1}{4} |\langle [\hat{A}_s, \hat{B}_s]_+ \rangle|^2$

which must also be true?

If $[\hat{A}, \hat{B}] = 0$ $\qquad \langle [\hat{A}_s, \hat{B}_s]_+ \rangle = \langle (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle) + (\hat{B} - \langle \hat{B} \rangle)(\hat{A} - \langle \hat{A} \rangle) \rangle$

$$\langle [\hat{A}_s, \hat{B}_s]_+ \rangle = \langle \hat{A}\hat{B} + \hat{B}\hat{A} - \langle \hat{A} \rangle \hat{B} - \hat{A} \cancel{\langle \hat{B} \rangle} + \cancel{\langle \hat{A} \rangle \langle \hat{B} \rangle} - \langle \hat{B} \rangle \hat{A} - \hat{B} \cancel{\langle \hat{A} \rangle} + \cancel{\langle \hat{B} \rangle \langle \hat{A} \rangle} \rangle$$

$$= 2\left(\langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \right)$$

$$\int \Psi^*(\vec{x},t)\, \hat{A}\, \hat{B}\, \Psi(x,t)\, d^3x = \langle \hat{A}\hat{B} \rangle$$

$$\left(\int \Psi^*(\vec{x},t)\, \hat{A}\, \Psi(\vec{x},t)\, d^3x \right)\left(\int \Psi^*(\vec{x},t)\, \hat{B}\, \Psi(\vec{x},t)\, d^3x \right) = \langle \hat{A} \rangle \langle \hat{B} \rangle$$

$$\Psi(\vec{x},t) = \sum_{a,b} c_{ab}\, \phi_{ab} \quad \ni \quad \hat{A}\phi_{ab} = a\phi_{ab} \qquad (\hat{A}\hat{B})\phi_{ab} = ab\, \phi_{ab}$$
$$\hat{B}\phi_{ab} = b\phi_{ab}$$

$$\langle \hat{A} \rangle = \int \sum_{a',b'} c_{a'b'}\, \phi^*_{a'b'}(\vec{x},t)\, \hat{A} \sum_{a,b} c_{ab}\, \phi_{ab}(\vec{x},t)\, d^3x$$

$$\underline{\langle \hat{A} \rangle} = \int \sum_{a',b'} C^*_{a'b'} \phi^*_{a'b'}(\vec{x},t)\, \hat{A} \sum_{a,b} C_{ab}\, \phi_{ab}(\vec{x},t)\, d^3x$$

$$= \sum_{a',b'} \sum_{a,b} \int \phi^*_{a'b'}(\vec{x},t)\, \hat{A}\, \underbrace{\phi_{ab}}(\vec{x},t)\, d^3x \; C^*_{a'b'}\, C_{ab}$$

$$= \sum_{a,b} |C_{ab}|^2\, a$$

$$\underline{\langle \hat{B} \rangle} = \sum_{a,b} |C_{ab}|^2\, b \quad , \quad \underline{\langle \hat{A}\hat{B} \rangle} = \sum_{a,b} |C_{ab}|^2\, a\, b$$

$$\langle \hat{A}^2 \rangle = \sum_{a,b} |C_{ab}|^2\, a^2$$

$$(\Delta \hat{A})^2 = \sum_{a,b} |C_{ab}|^2\, a^2 - \Big(\sum_{a,b} |C_{ab}|^2\, a \Big)^2 \neq 0 \quad \text{but} > 0$$

Example Infinite Potential Well

$$[\hat{H}, \hat{P}] = 0 \quad , \quad \psi_\pm = \frac{1}{\sqrt{2}} \left(e^{ikx} \pm e^{-ikx} \right)$$

$$\langle \hat{H} \rangle = E = \frac{\hbar^2 k^2}{2m} \quad ; \quad \langle \hat{H}^2 \rangle = E^2$$

If $[\hat{A}, \hat{B}]_- = 0 \Rightarrow \Delta\hat{A}\, \Delta\hat{B} \geq 0$ } must be satisfied always

In the particular case where $\Psi(\vec{x},t)$ is an eigenfunction of $\hat{A}$ then $\Delta\hat{A} = 0$, likewise for $\hat{B}$.

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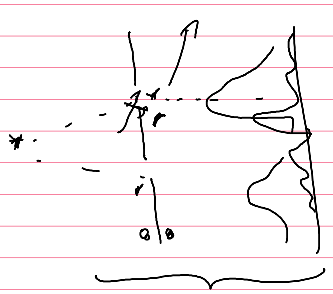
The Uncertainty Relation is related to the Superposition Principle.



$$\psi = a\psi_1 + b\psi_2 \Rightarrow \text{Uncert.}$$

Note $$(\Delta\hat{A})^2 (\Delta\hat{B})^2 = \frac{1}{2} \left| \langle [\hat{A}, \hat{B}]_- \rangle \right|^2$$

If 1) $\hat{A}_s \Psi(\bar{x},t) = c\,\hat{B}_s \Psi(\bar{x},t)$

2) $\langle [\hat{A}_s, \hat{B}_s]_+ \rangle = 0$