Class 9 The Minimum Uncertainty Product

Tuesday, September 18, 2018 8:01 AM

$$(\Delta\hat{A})\ (\Delta\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]_-\rangle\right|$$

$$(\Delta\hat{A})^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$$

For the Inequality to be saturated two conditions must be met:

1) $\varphi = c\psi \implies |\langle\varphi|\psi\rangle|^2 = \langle\varphi|\varphi\rangle\langle\psi|\psi\rangle$

$$\varphi \equiv \hat{A}_s\Psi(\bar{x},t) \quad ; \quad \psi \equiv \hat{B}_s\Psi(\bar{x},t)$$

$$\hat{A}\Psi(\bar{x},t) = c\ \hat{B}_s\Psi(\bar{x},t)$$

2) $\langle[\hat{A}_s,\hat{B}_s]_+\rangle = 0$

If both conditions are met then, $\Delta\hat{A}\Delta\hat{B} = \frac{1}{2}\langle[\hat{A},\hat{B}]_-\rangle$

<u>Example</u>: $\hat{A} = \hat{p}$, $\hat{B} = \hat{x}$

1) $\hat{p}_s\Psi(x,t) = c\ \hat{x}_s\Psi(x,t)$

$\implies$ 2) $\langle[\hat{x}_s,\hat{p}_s]\rangle = 0$ $\qquad \hat{x}_s = \hat{x} - \langle x\rangle$, $\hat{p}_s = \hat{p} - \langle\hat{p}\rangle$

These two conditions determine the general form for $\Psi(x,t)$

1) * $\left(\hat{p} - \langle\hat{p}\rangle\right)\Psi(x,t) = c\left(\hat{x} - \langle\hat{x}\rangle\right)\Psi(x,t)$ *

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$-i\hbar\frac{\partial}{\partial x}\Psi(x,t) = \left(cx - c\langle\hat{x}\rangle + \langle\hat{p}\rangle\right)\Psi(x,t)$$

$$\Psi(x,t) = \psi(x)\phi(t)$$

$$-i\hbar\frac{d}{dx}\psi(x) = (cx + \xi)\psi(x) \quad ; \quad \underline{\xi \equiv \langle\hat{p}\rangle - c\langle\hat{x}\rangle}$$

$$\int\frac{d\psi(x)}{\psi(x)} = \frac{i}{\hbar}\int(cx + \xi)dx$$

$$\int \frac{d\psi(x)}{\psi(x)} = \frac{i}{\hbar}\int (cx + \xi)\,dx$$

$$\ln \psi(x) = \frac{i}{\hbar}\left(\frac{cx^2}{2} + \xi x + \text{const}\right)$$

$$\psi(x) = A\, e^{i/\hbar\left(\frac{cx^2}{2} + \xi x\right)}$$

Claim: $c^* = -c$

Check: $\int \Psi^*(x,t)\,\hat{X}\left(\hat{P} - \langle \hat{P}\rangle\right)\Psi(x,t)\,dx = c\int \Psi^*(x,t)\,\hat{X}\left(\hat{X} - \langle \hat{X}\rangle\right)\Psi(x,t)\,dx$

$$* \quad \langle \hat{X}\hat{P}\rangle - \langle \hat{X}\rangle\langle \hat{P}\rangle = c\left(\langle \hat{X}^2\rangle - \langle \hat{X}\rangle^2\right)$$

$$\langle \hat{X}\hat{P}\rangle - \langle \hat{X}\rangle\langle \hat{P}\rangle = c\,(\Delta \hat{X})^2$$

$$\Rightarrow \langle \hat{X}\hat{P}\rangle^* - \langle \hat{X}\rangle\langle \hat{P}\rangle = c^*(\Delta \hat{X})^2$$

$$* \quad \langle \hat{P}\hat{X}\rangle - \langle \hat{X}\rangle\langle \hat{P}\rangle = c^*(\Delta \hat{X})^2$$

Adding the two:

$$\langle [\hat{X},\hat{P}]_+\rangle - 2\langle \hat{X}\rangle\langle \hat{P}\rangle = (c+c^*)(\Delta \hat{X})^2$$

<u>Note</u>:

$$[\hat{X}_s,\hat{P}_s]_+ = [\hat{X} - \langle \hat{X}\rangle, \hat{P} - \langle \hat{P}\rangle]_+$$

$$= (\hat{X} - \langle \hat{X}\rangle)(\hat{P} - \langle \hat{P}\rangle) + (\hat{P} - \langle \hat{P}\rangle)(\hat{X} - \langle \hat{X}\rangle)$$

$$= [\hat{X},\hat{P}]_+ - 2\left(\langle \hat{X}\rangle\hat{P} + \hat{X}\langle \hat{P}\rangle\right)$$

$$\Rightarrow \langle [\hat{X}_s,\hat{P}_s]_+\rangle = \underbrace{\langle [\hat{X},\hat{P}]_+\rangle - 2\langle \hat{X}\rangle\langle \hat{P}\rangle}$$

$$\therefore \quad \langle [\hat{X}_s,\hat{P}_s]_+\rangle = (c+c^*)(\Delta \hat{X})^2$$

$$= 0 \Rightarrow c + c^* = 0$$

$c = -c^*$ "pure imaginary"

$$c = i\,|c|$$

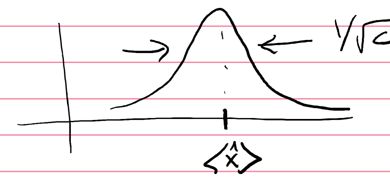
$$\therefore \quad \psi(x) = A\, e^{-\frac{|c|\,x^2}{2\hbar}}\, e^{\frac{i}{\hbar}\xi x} \qquad \xi = \langle\hat{p}\rangle - c\langle\hat{x}\rangle$$

$$\psi(x) = \psi(0)\, e^{-\frac{|c|\,x^2}{2\hbar}}\, e^{\frac{i}{\hbar}\langle\hat{p}\rangle x}\, e^{\frac{|c|}{\hbar}\langle\hat{x}\rangle x}$$

$$= \psi(0)\, e^{\frac{-|c|}{2\hbar}\left(x^2 - 2\langle\hat{x}\rangle x\right)}\, e^{\frac{i}{\hbar}\langle\hat{p}\rangle x}$$

$$= \psi(0)\, e^{\frac{-|c|}{2\hbar}\langle\hat{x}\rangle^2}\, e^{\frac{-|c|}{2\hbar}\left(x - \langle\hat{x}\rangle\right)^2}\, e^{i/\hbar\,\langle\hat{p}\rangle x}$$

$$|\psi(x)|^2 \propto e^{\frac{-|c|}{\hbar}\left(x - \langle\hat{x}\rangle\right)^2}$$



$\therefore$ A gaussian wavefunction will lead to

$$\boxed{\Delta\hat{x}\,\Delta\hat{p} = \hbar/2}$$

"saturates the uncertainty relation"

For a free particle : $\hat{H} = \dfrac{\hat{p}^2}{2m}$

The Eigenfunction of $\hat{H}$: $\underbrace{\psi(x) = A\, e^{\pm\frac{i}{\hbar}px}} \Rightarrow E = \dfrac{p^2}{2m}$

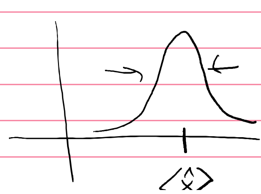
$$\Delta p = \left[\langle p^2\rangle - \langle p\rangle^2\right]^{1/2}$$

$$\langle p^2\rangle = \int \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)^2\psi(x)\,dx = p^2\underbrace{\int\psi^*(x)\,\psi(x)\,dx}$$

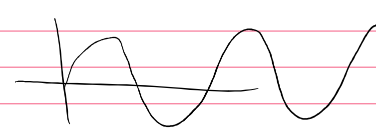
$$\langle p\rangle = p\int\psi^*(x)\,\psi(x)\,dx \qquad p\psi = p\psi$$

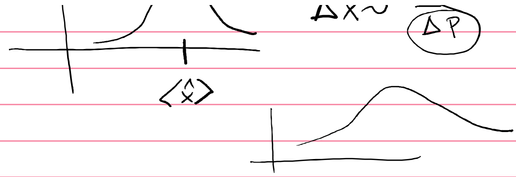
$$\begin{cases} \Delta p = 0 \\ \Delta x \to \infty \end{cases} \qquad \Delta x\,\Delta p \quad \text{"not well defined"}$$

$$\Delta x\,\Delta p \geq \hbar/2 \Rightarrow \Delta x \sim \frac{\hbar}{2}\frac{1}{\Delta p}$$



$\Delta x \sim \dfrac{1}{\Delta p}$





$\Delta x \sim \Delta P$

$(\Delta X)_t^2 \sim t^2$ $\qquad$ $(\Delta X)_t^2 \, (\Delta P)_t^2 \sim t^2$ $\qquad$ $\hbar/2$

$(\Delta P)_t^2 \sim (\Delta P)_0$ $\qquad$ $(\Delta x)_t (\Delta P)_t \sim t + (\Delta x)_0 (\Delta P)_0$

$$(\Delta X)_0 \, (\Delta P)_0 = \hbar/2 \quad \text{for} \quad \psi(x) \sim e^{-\frac{|c|}{2\hbar} x^2}$$

$$[X, P] = i\hbar$$

For Homework you found:

$$(\Delta \hat{X})^2 = (\Delta \hat{X})_0^2 + \frac{t}{m} \langle \hat{X}\hat{P} + \hat{P}\hat{X} \rangle_0 + \frac{(\Delta \hat{P})_0^2 \, t^2}{m^2} - \frac{2 \langle \hat{X} \rangle_0 \langle \hat{P} \rangle_0 \, t}{m} \qquad *$$

For gaussian $(\Delta \hat{X})_0 (\Delta \hat{P})_0 = \frac{\hbar}{2}$

$$(\Delta \hat{P})_t = (\Delta \hat{P})_0 \qquad \text{"for free Particle"}$$

$$* \quad \langle [\hat{X}, \hat{P}] \rangle_0 = \underbrace{\langle [\hat{X}_s, \hat{P}_s] \rangle_0}_{0} + 2 \langle \hat{X} \rangle_0 \langle \hat{P} \rangle_0$$

Then $$(\Delta \hat{X})^2 = (\Delta x)_0^2 + \left(\frac{\hbar}{2} \frac{1}{\Delta X_0} \right)^2 \frac{t^2}{m^2}$$

$$= (\Delta x)_0^2 \left[1 + \left(\frac{\hbar t}{2m (\Delta X)_0^2} \right)^2 \right] \qquad *$$

Further comment about the relation between the Superposition Principle & the Uncertainty Relation:

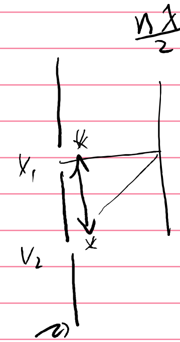
We cannot have simultaneous eigenfunctions of X & P, but suppose we have eigenfunction of P: $\hat{p}\,\psi(x) = p\,\psi(x)$

$\hat{X}\,\varphi(x_i) = x_i \varphi_i(x)$

Suppose we have eigenfunctions of $\hat{x}$: $\quad \hat{p}\,\phi(x) = p\,\phi(x)$

$$\hat{x}\,\phi(x) = x_i\,\phi_i(x)$$

$$\psi(x) = \sum_i c_i\,\phi_i(x) \qquad \Delta\hat{p} = 0$$



$$= c_1\phi_1(x) + c_2\phi_2(x)$$

Suppose only two as in the double slit exp.

$$\langle x \rangle = \int \left(c_1^*\phi_1 + c_2^*\phi_2\right)\left(\hat{x}\right)\left(c_1\phi_1 + c_2\phi_2\right) dx$$

$$= \int \Big[\left(|c_1|^2 x_1 |\phi_1|^2 + |c_2|^2 x_2 |\phi_2|^2 + c_1^* c_2\, \phi_1^* \phi_2\, x_2 + c_2^* c_1\, \phi_2^* \phi_1\, x_1\right)\Big]\, dx$$

$$(\Delta x) \Rightarrow \infty$$

The Uncertainty Relation Imposes Lower limits on the allowed energy of bound systems:

$$\hat{H} = \frac{p^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$\Delta x \sim a \qquad \Delta p \gtrsim \frac{\hbar}{2a} \sim p$$

$$\Rightarrow E = \frac{\hbar^2}{4a^2(2m)} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{a}$$

To minimize E:

$$\frac{\partial E}{\partial a} = 0 \quad \Rightarrow \quad a = \frac{\hbar^2\, 4\pi\epsilon_0}{m_e e^2}$$

"Bohr Radius"

$$\Rightarrow E = -13.6\ \text{eV}$$