

For  $H \neq H(t)$  the solns to the TDSE:

$$\Psi(\vec{x}, t) = e^{-i\hat{H}t/\hbar} \Psi(\vec{x}, 0)$$

also  $\Psi(\vec{x}, 0) = \sum_n c_n \psi_n(x)$

$$\hat{H} \psi_n(x) = E_n \psi_n(x)$$

General Properties of the solutions of the TISE:

1) Orthogonality:  $\int \psi_m^*(\vec{x}) \psi_n(\vec{x}) d^3x = \delta_{mn}$

2) Reality: If  $\psi_n(x)$  is a soln then so is  $\psi_n^*(x)$

3) There is always a lower bound on the allowed energy eigenvalue.

$$E_0 > V_{\min}$$

↳ lowest allowed energy

Proof:  $\langle \hat{H} \rangle = \int \Psi^*(\vec{x}, t) \hat{H} \Psi(\vec{x}, t) d^3x$

$$= \int \Psi^*(\vec{x}, 0) \hat{H} \Psi(\vec{x}, 0) d^3x$$

If  $\Psi(\vec{x}, 0) = \psi(\vec{x})$  "Superposed state"

$$\langle \hat{H} \rangle = \int \psi^*(\vec{x}) \hat{H} \psi(\vec{x}) d^3x$$

$$= \int \psi^*(\vec{x}) \left( \frac{\hat{p}^2}{2m} + V(\vec{x}) \right) \psi(\vec{x}) d^3x$$

$$= \frac{1}{2m} \int (\hat{p}\psi(\vec{x}))^* (\hat{p}\psi(\vec{x})) d^3x + \int \psi^*(\vec{x}) V(\vec{x}) \psi(\vec{x}) d^3x$$

$$\underbrace{\int |\psi'(\vec{x})|^2 d^3x}_{\text{positive definite}} \quad \text{where } \psi'(\vec{x}) = \hat{p}\psi(\vec{x})$$

$$\langle \hat{H} \rangle \geq \int \psi^*(\vec{x}) V(\vec{x}) \psi(\vec{x}) d^3x$$

$V(\vec{x})$

$$\int |\psi(\vec{x})|^2 V(\vec{x}) d^3x \geq \int |\psi(\vec{x})|^2 V_{\min} d^3x$$

$$\langle \hat{H} \rangle > V_{\min}$$

Recall  $\langle \hat{H} \rangle = \sum_n |c_n|^2 E_n > V_{\min}$

"for Any Superposed state"

In the case where  $\psi(\vec{x}) = \psi_0(\vec{x})$   
 $\hookrightarrow$  the lowest energy state.

$$\Rightarrow \langle \hat{H} \rangle = E_0$$

$$E_0 > V_{\min}$$

Example:  $V(x) = \frac{1}{2} kx^2$      $V_{\min} = 0 \Rightarrow E_0 > 0$

4) Parity if  $V(-x) = V(x)$  then the solutions can be separated into even & odd.

5) The solutions to TISE allow for both bound state & un-bound state solutions:

for simplicity we will work in one dimension:

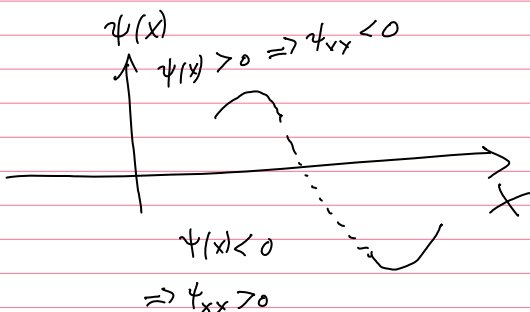
$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x)$$

$$\psi_{xx}(x) = \frac{d^2 \psi(x)}{dx^2} = \frac{2m}{\hbar^2} (V(x) - E) \psi(x)$$

If  $E > V(x)$ :

$$\psi_{xx}(x) = -\frac{2m}{\hbar^2} (E - V(x)) \psi(x)$$

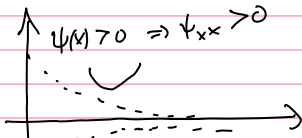
$$= -k^2 \psi(x) ; \quad k^2 \geq 0$$



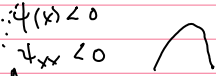
$\Rightarrow$  solutions oscillate  
 "Unbound state solutions"

If  $E < V(x)$

$$\psi_{xx}(x) = \kappa^2 \psi(x) \quad \kappa^2 \geq 0$$



"non-oscillatory solution"



classically prohibited

$$E = T + V$$

