Class 10 The time uncertainty relation and Particle Decays

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$$\Delta P \, \Delta X \geq \frac{\hbar}{2} \quad ; \qquad \Delta \hat{A} \, \Delta \hat{B} \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

From this relation one can obtain an "uncertainty-like" relation for energy and time.

From Relativistic Mechanics : $\vec{P} = \gamma m \vec{V}$ $\qquad \gamma = \dfrac{1}{\sqrt{1 - V^2/c^2}}$

$$E = \gamma m c^2$$

$$E^2 = p^2 c^2 + m^2 c^4 \quad ; \quad cP/E = V/c \Rightarrow \frac{E}{cP} = \frac{c}{V}$$

$$2 E \, dE = 2 p \, dP \, c^2$$

$$\Rightarrow dp = E/pc^2 \, dE \Rightarrow \Delta P = E/pc^2 \, \Delta E$$

$$= 1/V \, \Delta E = \frac{\Delta t}{\Delta x} \Delta E$$

$$\Delta P \, \Delta X = \Delta t \, \Delta E$$

$$\Delta E \, \Delta t \geq \hbar/2$$

$$\Delta t \neq \cancel{\langle t^2 \rangle - \langle t \rangle^2}$$

Recall,

$$\frac{d\langle \hat{Q} \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle \qquad \text{assuming } \frac{\partial \hat{Q}}{\partial t} = 0$$

$$[\hat{H}, \hat{Q}]_- \neq 0$$

$$\text{but } \frac{1}{2} \left| \langle [\hat{H}, \hat{Q}] \rangle \right| \leq \Delta \hat{H} \, \Delta \hat{Q}$$

$$\Delta \hat{H} \, \Delta \hat{Q} \geq \frac{\hbar}{2} \left| \frac{d\langle \hat{Q} \rangle}{dt} \right|$$

Defn

$$\Delta t \equiv \frac{\Delta \hat{Q}}{\left| \frac{d\langle \hat{Q} \rangle}{dt} \right|}$$

$$\Delta \hat{H} \, \Delta t \geq \frac{\hbar}{2}$$

$$\Delta E \, \Delta t \geq \hbar/2$$

Δt = time that it takes the expectation value of $\hat{Q}$

to change by one std deviation.

If any observable of the system changes rapidly $\Rightarrow$ ΔE must be large.

Unstable Particle: In the particles rest frame $\Delta E \sim Mc^2$

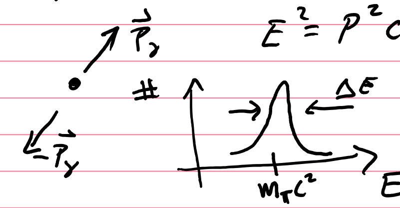
$\Delta E\, \Delta t \geq \hbar/2$

$$\Delta E\, \Delta t \sim \frac{\hbar}{2}$$

$$Mc^2 \Delta t \sim \hbar/2 \qquad \Delta t \sim \tau \quad \text{"lifetime"}$$

$$\tau \sim \frac{\hbar}{2Mc^2}$$

$$\pi^0 \to \gamma\gamma \qquad E^2 = p^2c^2$$



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Stationary States: Eigenstates of $\hat{H}$

$$\hat{H}\psi(x) = E\psi(x) \qquad \Psi(\vec{x},t) = e^{-i/\hbar\, \hat{H} t}\, \Psi(x,0)$$

If $\Psi(x,0)$ is an eigenfunction of $\hat{H}$, then,

$$\Psi(\vec{x},t) = e^{-i/\hbar\, E t}\, \Psi(\vec{x},0) = e^{-i/\hbar\, E_n t}\, \psi_n(\vec{x})$$

In contrast a wave function which is not an eigenfunction of $\hat{H}$ can always be expanded as a linear superposition of eigenfunctions of $\hat{H}$, i.e.

$$\Psi(\vec{x},0) = \sum_n\!\!\!\!\!\!\int\; c_n \psi_n(\vec{x}) \qquad \text{where} \quad \hat{H}\psi_n(\vec{x}) = E_n \psi_n(\vec{x})$$

↳ "Superposed State"

The time evolution of a superposed state is a bit more complicated,

$$\Psi(\vec{x},t) = e^{-i/\hbar\, \hat{H} t}\, \Psi(\vec{x},0)$$

$$= \sum_n c_n\, e^{-i/\hbar\, E_n t}\, \psi_n(\vec{x})$$

$$\langle \hat{A} \rangle = \sum c_{n'}^* \; e^{-i/\hbar (E_n - E_{n'}) t} \; \langle \psi_{n'}(\vec{x}) | \hat{A} | \psi_n(\vec{x}) \rangle$$

$$\langle A \rangle_{superp} = \sum_{n'} c_n$$

$$\langle \hat{A} \rangle_{stationary} = e^{\frac{i}{\hbar} E_n t} e^{-i/\hbar E_n t} \langle \psi_n(\vec{x}) | \hat{A} | \psi_n(\vec{x}) \rangle$$

$$= \langle \psi_n(\vec{x}) | \hat{A} | \psi_n(\vec{x}) \rangle$$ "No time dependence"

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$$\frac{d\langle \hat{A} \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle$$

for an stationary state:

$$= \frac{i}{\hbar} \int d^3x \; \psi_n^*(\vec{x}) \left[\hat{H}\hat{A} - \hat{A}\hat{H} \right] \psi_n(\vec{x})$$

$$= \frac{i}{\hbar} \left[\int d^3x \left(\hat{H} \psi_n(\vec{x}) \right)^* \hat{A} \psi_n(\vec{x}) - \int d^3x \; \psi_n^*(\vec{x}) \hat{A} \hat{H} \psi_n(x) \right]$$

$$= \frac{i}{\hbar} \left[E_n \int d^3x \; \psi_n^*(\vec{x}) \hat{A} \psi_n(\vec{x}) - E_n \int d^3x \; \psi_n^*(\vec{x}) \hat{A} \psi_n(\vec{x}) \right]$$

$$= 0$$

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for stationary states $\frac{d\langle \hat{Q} \rangle}{dt} = 0 \Rightarrow \Delta t \rightarrow 0$

$\Rightarrow$ we can have $\Delta E = 0$

$$(\Delta E)^2 = \langle H^2 \rangle - \langle \hat{H} \rangle^2 = \begin{cases} E^2 - E^2 = 0 & \text{for stationary states} \\ & \end{cases}$$

But $\frac{d\langle \hat{Q} \rangle}{dt} = 0$ also when $[\hat{H}, \hat{Q}] = 0$

so again $\Delta E = 0$ is allowed.

$$\langle \hat{Q} \rangle = \int \Psi^*(\vec{x}, t) \; \hat{Q} \; \Psi(x, t) \; d^3x$$

$$\Psi(\vec{x}, t) = \sum_{n, q} c_{nq} e^{-i/\hbar E_n t} \psi_{nq}(\vec{x})$$

$$\hat{Q} \psi_{nq} = q \psi_{nq}$$

$$\hat{H} \psi_{nq} = E_n \psi_{nq}$$

$$(\Delta E)^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$$

$$(\Delta E)^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$$

$$\stackrel{?}{=} 0 \qquad \langle H^2 \rangle = \int \Psi^*(\vec{x},t)\, \hat{H}^2\, \Psi(\vec{x},t)\, d^3x$$

$$\langle H \rangle = \int \Psi^*(\vec{x},t)\, \hat{H}\, \Psi(\vec{x},t)\, d^3x$$

Example : $\frac{d\langle \hat{Q} \rangle}{dt} = 0$ for stationary state & $\Delta E = 0$

Consider a two state system

$$\Psi(x,t) = \frac{e^{-i/\hbar E_1 t}\psi_1(x) + e^{-i/\hbar E_2 t}\psi_2(x)}{\sqrt{2}}$$

$$\hat{H}\psi_i(x) = E_i\,\psi_i(x)$$

$$i = 1, 2$$

$$\frac{d\langle \hat{Q} \rangle}{dt} = \int \Psi^*(\vec{x},t)\, \hat{Q}\, \Psi(x,t)\, dx$$

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$$\frac{d\langle \hat{Q} \rangle}{dt} = (E_1 + E_2)\, \hat{Q}_{12} \cos \omega_{21} t$$

$$\hat{Q}_{12} \equiv \int \psi_1^*(x)\, \hat{Q}\, \psi_2(x)\, dx$$

$$\omega_{21} = \frac{E_2 - E_1}{\hbar} \quad ; \text{Assumed } \hat{Q}_{12}^* = \hat{Q}_{12}$$

$\therefore$ Here $\Delta t \sim$ Period of oscillation

$$\Delta t \sim T = \frac{2\pi}{|\omega_{21}|} = \frac{2\pi}{|E_2 - E_1|}\hbar = \frac{\pi\hbar}{\Delta E} \qquad \Delta E = \frac{|E_2 - E_1|}{2}$$

$$\Delta E \Delta t \sim \pi\hbar > \hbar/2$$

$$\Delta E = \frac{|E_1 - E_2|}{2} \quad *$$